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Some fixed point theorems for generalized $\mathcal{R}$-Lipschitz mappings in linear cone 2-normed spaces

Abstract. In this paper, we introduce the concept of linear cone 2-normed spaces and prove some fixed point results for generalized $\mathcal{R}$-Lipschitz contractions in linear cone 2-normed spaces endowed with a binary relation $\mathcal{R}$. We observe that the fixed point of the considered mappings can be approximated with Mann iteration scheme. Our results generalize and extend several known results of literature into linear cone 2-normed spaces. Some examples are provided which illustrate the new concepts and the results.

Keywords. Cone 2-normed space, binary relation, generalized $\mathcal{R}$-Lipschitz mapping, fixed point.

Mathematics Subject Classification (2010): 47H10; 54H25.

1 - Introduction and Preliminaries

Fréchet [**7**] introduced the notion of metric spaces. The metric function was defined from the product $X \times X$ into the nonnegative real numbers, where X is a nonempty set. The concept of 2-metric spaces was introduced by Gähler [**8**, **9**, **10**]. Gähler defined the metric function from the product $X \times X \times X$ into the nonnegative real numbers, and improved the notion of usual metric. The concept of 2-normed spaces was introduced by Gähler [**8**] as a generalization of a normed linear spaces. On the other hand, in 1934, Kurepa [**16**] (see also, [**17**]) introduced some abstract metric spaces in which the metric function was defined from the product $X \times X$ into an ordered vector space. In 2007, Huang and Zhang [**12**] reintroduced such spaces and called it cone metric spaces. They also proved some fixed point results in such spaces with exploiting the notion of normality

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of a cone. In view of some recent papers [**3**, **5**, **6**, **13**] we notice that the fixed point results in cone metric spaces are a consequence of their corresponding metric versions, therefore are equivalent to those in metric spaces. Li and Xu [**18**] reconsidered the notion of cone metric and define the cone metric spaces over Banach algebra. They also improved the contractive conditions on self-mappings used by Huang and Zhang [**12**] by using a vector contractive constant instead a real number. They also showed by an example that their results cannot be obtained as a consequence of corresponding usual metric versions.

Recently, Singh et al. [**25**] introduced the notion of cone 2-metric spaces by combining the notions of 2-metric and cone metric spaces and prove a fixed point result for the mappings satisfying some contractive conditions. Following the lines of Singh et al. [**25**] and Li and Xu [**18**], Wang et al. [**29**] improved the result of Singh et al. [**25**] by introducing the cone 2-metric spaces over Banach algebra. They also improved the contractive conditions by introducing the vector contractive constants.

Ran and Reurings [**22**] proved some fixed point results in metric spaces endowed with partial order and discussed some applications of their results. Results of Ran and Reurings [**22**] were discussed and generalized by several authors in different directions, see, e.g., [**1**, **2**, **19**, **20**, **26**, **27**, **28**]. In this paper, we introduce the notion of cone 2-normed spaces over Banach algebra which extend and generalize the notions of norm and 2-norm spaces. Some fixed point results for generalized $\mathcal{R}$-Lipschitz contractions on cone 2-normed spaces over Banach algebra endowed with a binary relation $\mathcal{R}$ are proved, which generalize several known results into this new setting. Also, we approximate the fixed point of the considered mappings with Mann iteration scheme. New concepts and results are illustrated by some examples.

First, we recall some definitions and properties which will be used in the sequel.

In the further discussion, we always suppose that A is a Banach algebra with multiplicative unit e, that is, $ex = xe = x$ for all $x \in A$. The following proposition can be found, e.g., in [**23**].

Proposition 1.1. *Let A be a Banach algebra with the unit e and $x \in A$. If the spectral radius $\rho(x)$ of x is less than 1, that is,*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n} < 1$$

then $e - x$ is invertible. Actually, $(e-x)^{-1} = \sum_{i=0}^{\infty} x^i$. It is obvious that if $\rho(x) < 1$, then $\|x^n\| \to 0$ as $n \to \infty$.

A subset P of A is called a cone if:

(1) P is nonempty closed and $\{\theta, e\} \subset P$;

(2) $\alpha P + \beta P \subset P$ for all nonnegative real numbers α, β;

(3) $P^2 = PP \subset P$;

(4) $P \cap (-P) = \{\theta\}$

where θ and e are respectively the zero vector and unit of A.

Given a cone $P \subset A$, we define a partial ordering $\preceq$ in A with respect to P by $x \preceq y$ (or equivalently $y \succeq x$) if and only if $y - x \in P$. We shall write $x \prec y$ (or equivalently $y \succ x$) to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ (or equivalently $y \gg x$) will stand for $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P.

The cone is called normal if there exists a number $K > 0$ such that for all $x, y \in P$

$$x \preceq y \implies \|x\| \leq K\|y\|.$$

The least number K satisfying the above inequality is called the normal constant of P. The cone P is called solid if $\mathrm{int}P \neq \emptyset$.

In the following, we always assume that the cone P is solid in Banach algebra A and $\preceq$ is the partial ordering with respect to P.

Proposition 1.2 ([**30**]). *Let A be a Banach algebra with the unit e, P be a cone in A. Then, for any $a, b \in A$, $c \in P$ with $a \preceq b$ we have $ac \preceq bc$.*

Lemma 1.3 ([**15**, **21**]). *Let A be a Banach algebra with a solid cone P. Then:*

(a) *If $a \preceq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

(b) *If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

(c) *If $\|x_n\| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $x_n \ll c$ for all $n > n_0$.*

Definition 1.4 ([**4**, **14**]). Let P be a solid cone in a Banach algebra A. A sequence $\{u_n\} \subset P$ is a c-sequence if for each $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n > n_0$.

Proposition 1.5 ([**30**]). *Let P be a solid cone in a Banach algebra A and let $\{u_n\}$ be a sequence in P. Suppose that $a \in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in P. Then $\{au_n\}$ is a c-sequence.*

Proposition 1.6 ([**14**]). *Let P be a solid cone in a Banach algebra A and let $\{u_n\}$, $\{v_n\}$ be two sequences in P. If $\{u_n\}$ and $\{v_n\}$ are c-sequences in P, $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence.*

L e m m a 1.7 ([**30**]). *Let A be a Banach algebra and x, y be vectors in A. If x and y commute, then the following hold:*

(i) $\rho(xy) \leq \rho(x)\rho(y)$;

(ii) $\rho(x + y) \leq \rho(x) + \rho(y)$;

(iii) $|\rho(x) - \rho(y)| \leq \rho(x - y)$.

L e m m a 1.8 ([**24**]). *Let A be a Banach algebra with unit e, P be a cone in A and $a, b, c \in P$. Then:*

(i) *If $\rho(a) < 1$, then $\rho(a^m) \leq \rho(a) < 1$ for each $m \in \mathbb{N}$.*

(ii) *If $\rho(a) < 1$ and $b \preceq ac$, then $b \preceq c$.*

2 - Linear cone 2-normed space

In this section, we define the linear cone 2-normed spaces and some properties of such spaces.

D e f i n i t i o n 2.1. Let X be a linear space over a field $\mathcal{F}$ ($\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$) with $\dim X \geq 2$. Let A be a Banach algebra, P a solid cone in A and $\|\cdot,\cdot\|\colon X \times X \to P$ be a function. Then $(X, \|\cdot,\cdot\|)$ is called a linear cone 2-normed space over Banach algebra A if the following conditions are satisfied:

(a) $\|x, y\| = \theta$, if and only if x and y are linearly dependent;

(b) $\|x, y\| = \|y, x\|$;

(c) $\|\alpha x, y\| = |\alpha|\|x, y\|$;

(d) $\|x, y + z\| \preceq \|x, y\| + \|x, z\|$

for all $x, y, z \in X$ and $\alpha \in \mathcal{F}$. The space $(X, \|\cdot,\cdot\|)$ is called normal, if P is a normal cone.

E x a m p l e 2.2. Let $X = \mathbb{R}^2$, $A = C^1_{\mathbb{R}}[0, 1]$ with point-wise multiplication and norm defined by $\|x(t)\| = \|x(t)\|_\infty + \|x'(t)\|_\infty$. Let $P = \{\psi(t) \in A\colon \psi(t) \geq 0 \text{ for all } t \in [0, 1]\}$ be the solid cone in A and define a mapping $\|\cdot,\cdot\|\colon X \times X \to P$ by

$$\|c_1, c_2\| = \left|\det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right| e^t$$

for all $c_1 = (x_1, x_2), c_2 = (y_1, y_2) \in X$. Then, $(X, \|\cdot,\cdot\|)$ is a linear cone 2-normed space.

Example 2.3. Let $X = \mathbb{R}^3$, $A = \mathbb{R}^2$ with multiplication defined by $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$ and norm defined by $\|(x_1, x_2)\| = |x_1| + |x_2|$. Let $P = \{(x_1, x_2) \colon x_1, x_2 \geq 0\}$ and define a mapping $\|\cdot,\cdot\| \colon X \times X \to P$ by $\|v_1, v_2\| = m(1, \alpha)$ for all $v_1 = (x_1, x_2, x_3), v_2 = (y_1, y_2, y_3) \in X$, where $\alpha > 0$ is fixed and

$$m = \left|\det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right| + \left|\det\begin{pmatrix} x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}\right| + \left|\det\begin{pmatrix} x_3 & y_3 \\ x_1 & y_1 \end{pmatrix}\right|.$$

Then, $(X, \|\cdot,\cdot\|)$ is a linear cone 2-normed space.

Lemma 2.4. *Let $(X, \|\cdot,\cdot\|)$ be a linear cone 2-normed space. If $x \in X$ is fixed and $\|x, y\| = \theta$ for all $y \in X$ then $x = 0_X$, where 0_X is the zero vector of X.*

Proof. Suppose $x \in X$ be fixed and $\|x, y\| = \theta$ for all $y \in X$. Then x and y are linearly dependent for all $y \in X$, therefore, we must have $x = 0_X$. □

Definition 2.5. Let $(X, \|\cdot,\cdot\|)$ be a linear cone 2-normed space. A sequence $\{x_n\}$ in X is called convergent if there exist $x \in X$ and $n_0 \in \mathbb{N}$ such that for every $c \in A$ with $\theta \ll c$ we have $\|x_n - x, z\| \ll c$ for all $z \in X$ and $n > n_0$. In other words, the sequence $\{x_n\}$ in X is called convergent and converges to $x \in X$ if $\{\|x_n - x, z\|\}$ is a c-sequence for all $z \in X$. In this case x is called the limit of sequence $\{x_n\}$ and we write $x_n \to x$ as $n \to \infty$.

Definition 2.6. Let $(X, \|\cdot,\cdot\|)$ be a linear cone 2-normed space. A sequence $\{x_n\}$ in X is called Cauchy if there exists $n_0 \in \mathbb{N}$ such that for every $c \in A$ with $\theta \ll c$ we have $\|x_n - x_m, z\| \ll c$ for all $z \in X$ and $n, m > n_0$.

Remark 2.7. The limit of a convergent sequence in a linear cone 2-normed space is unique.

Proof. Suppose the sequence $\{x_n\}$ has two distinct limits $x, y \in X$. Then, for every $c \in A$ with $\theta \ll c$ we have $\|x_n - x, z\| \ll c/2$ and $\|x_n - y, z\| \ll c/2$ for all $z \in X$ and $n > n_0$. Therefore:

$$\|x - y, z\| = \|x - x_n + x_n - y, z\| \preceq \|x_n - x, z\| + \|x_n - y, z\| \ll c/2 + c/2 = c$$

for all $z \in X$ and $n > n_0$. Therefore, by Lemma 1.3 we have $\|x - y, z\| = \theta$ for all $z \in X$, which with Lemma 2.4 yields $x - y = 0_X$, i.e., $x = y$. This contradiction proves the result. □

Definition 2.8. Let $(X, \|\cdot,\cdot\|)$ be a linear cone 2-normed space over Banach algebra A. Then, X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 2.9. Let $(X, \|\cdot,\cdot\|)$ be a linear cone 2-normed space over Banach algebra A and $T: X \to X$ be a mapping. Then, T is said to be continuous at a point $x \in X$ if for every sequence $\{x_n\} \subset X$ such that $x_n \to x$ as $n \to \infty$ we have $Tx_n \to Tx$ as $n \to \infty$. If the mapping T is continuous at every point of X then T is called a continuous mapping.

Let $(X, \|\cdot,\cdot\|)$ be a linear cone 2-normed space over Banach algebra A and $T: X \to X$ be a mapping. Then, the sequence $\{x_n\}$ defined by $x_n = T^n x_0$, where $T^0 x_0 = x_0$ is called the Picard sequence generated by T with initial value x_0, or the Picard iteration scheme with initial value x_0.

Suppose, X is a real linear space and C be a convex subset of X, $T: C \to C$ and $x_0 \in C$. The Mann iteration scheme with initial value x_0 is defined as follows:

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n T x_n$$

where $\beta_n \in (0, 1]$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, are such that $\sum_{i=0}^{\infty} \beta_i = \infty$.

Let X be a nonempty set and $T: X \to X$ be a mapping. Then by $O(T, x_0)$ we denote the set $\{T^n x_0 : n \in \mathbb{N}_0\}$, where $T^0 x_0 = x_0$ and it is called the orbit of T at x_0. The set of all fixed points of T is denoted by $\text{Fix}(T)$, i.e., $\text{Fix}(T) = \{x \in X : Tx = x\}$.

Definition 2.10. Let $(X, \|\cdot,\cdot\|)$ be a linear cone 2-normed space, $T: X \to X$ a mapping and $\mathcal{R}$ be a binary relation on X. Then, X is called $\mathcal{R}$-T-orbitally complete if every Cauchy sequence $\{x_n\} \subset O(T, x_0)$ with $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$ is convergent in X for all $x_0 \in X$.

It is obvious that every complete linear cone 2-normed space is $\mathcal{R}$-T-orbitally complete, but the converse is not true in general.

Example 2.11. Let $X = \mathbb{Q}^2$ be the linear space over the field of rational numbers, $A = C^1_{\mathbb{R}}[0, 1]$ with point-wise multiplication and norm defined by $\|x(t)\| = \|x(t)\|_\infty + \|x'(t)\|_\infty$. Let $P = \{\psi(t) \in A : \psi(t) \geq 0 \text{ for all } t \in [0, 1]\}$ be the solid cone in A and define a mapping $\|\cdot,\cdot\|: X \times X \to P$ by:

$$\|u, v\| = \begin{cases} 0, & \text{if } u = 0_X \text{ or } v = 0_X \text{ or } u = qv,\ q \in \mathbb{Q}; \\ x_m y_m e^t, & \text{otherwise} \end{cases}$$

for all $u = (x_1, x_2), v = (y_1, y_2) \in X$, where $x_m = \max\{|x_1|, |x_2|\}$ and $y_m = \max\{|y_1|, |y_2|\}$. Then $(X, \|\cdot,\cdot\|)$ is a linear cone 2-normed space over the Banach algebra. Define a subset $B \subset \mathbb{Q}$ by $B = \left\{\frac{p}{q} : p + q \text{ is even}\right\}$. Define a mapping $T\colon X \to X$ and a relation $\mathcal{R}$ on X by:

$$T\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) = \begin{cases} \frac{1}{3}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right), & \text{if } \frac{p_1}{q_1}, \frac{p_2}{q_2} \in B; \\ 2\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right), & \text{otherwise} \end{cases}$$

and $\mathcal{R} = \{(x_1, x_2), (y_1, y_2) \in X \times X : x_1, x_2, y_1, y_2 \in B\}$. Then it is easy to see that X is a $\mathcal{R}$-T-orbitally complete cone 2-normed space over Banach algebra A. On the other hand, X is not complete.

Let X be a nonempty set and $\mathcal{R}$ be a binary relation on X. Then, by $\mathcal{R}^{-1}$ and $\overline{\mathcal{R}}$ we denote the binary relations on X defined by $\mathcal{R}^{-1} = \{(x, y) \in X \times X : (y, x) \in \mathcal{R}\}$ and $\overline{\mathcal{R}} = \mathcal{R} \cup \mathcal{R}^{-1}$. The relation $\mathcal{R}^{-1}$ is called the reverse relation of $\mathcal{R}$. Notice that $\overline{\mathcal{R}}$ is symmetric and called the symmetric relation corresponding to $\mathcal{R}$. Let $A, B \subset X$, then we write $A \asymp B$ if $(a, b) \in \overline{\mathcal{R}}$ for all $a, b \in A \times B$. If $(X, \|\cdot,\cdot\|)$ be a linear cone 2-normed space, then the subset $A \subset X$ is called closed if for every sequence $\{x_n\} \subset A$ converging to some $x \in X$ we have $x \in A$. For a mapping $T\colon A \to A$ the subset $A \subset X$ is called $\mathcal{R}$-T-orbitally closed if for every $x_0 \in A$ and sequence $\{x_n\} \subset O(T, x_0)$ such that $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, we have $x \in A$.

3 - Fixed point theorems

In this section, we give some notions and prove some fixed point theorems for generalized $\mathcal{R}$-Lipschitz mapping in $\mathcal{R}$-T-orbitally complete linear cone 2-normed spaces over some Banach algebra and the fixed point is approximated by the Mann iteration scheme.

Definition 3.1. Let $(X, \|\cdot,\cdot\|)$ be a linear cone 2-normed space, $T\colon X \to X$ a mapping and $\mathcal{R}$ be a binary relation on X. The mapping T is called $\mathcal{R}$-preserving with respect to $\|\cdot,\cdot\|$ if:

$$(x, y) \in \mathcal{R},\ \|Tx - Ty, z\| \preceq \|x - y, z\|\ \forall z \in X \implies (Tx, Ty) \in \mathcal{R}.$$

Definition 3.2. Let $(X, \|\cdot,\cdot\|)$ be a linear cone 2-normed space over Banach algebra A, $T\colon X \to X$ be a mapping and $\mathcal{R}$ be a binary relation on X. Then T is said to be a generalized $\mathcal{R}$-Lipschitz contraction if there exists $k \in P$ such that $\rho(k) < 1$ and

$$\|Tx - Ty, z\| \preceq k\|x - y, z\| \tag{1}$$

for all $x, y, z \in X$ with $(x, y) \in \mathcal{R}$.

Remark 3.3. By the definition of $\overline{\mathcal{R}}$ and (1) it is clear that, T is a generalized $\mathcal{R}$-Lipschitz contraction if and only if T is a generalized $\overline{\mathcal{R}}$-Lipschitz contraction.

Theorem 3.4. *Let $(X, \|\cdot,\cdot\|)$ be a linear cone 2-normed space over Banach algebra A, $\mathcal{R}$ a binary relation on X and $T\colon X \to X$ be a generalized $\mathcal{R}$-Lipschitz contraction such that X is $\mathcal{R}$-T-orbitally complete. Suppose the following conditions are satisfied:*

(i) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in \mathcal{R}$;*

(ii) *T is $\mathcal{R}$-preserving with respect to $\|\cdot,\cdot\|$;*

(iii) *one of the following conditions is satisfied:*

  (a) *T is continuous;*

  (b) *if $\{x_n\}$ is the Picard sequence generated by T with initial value x_0, $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n_j}\}$ such that $(x_{n_j}, x) \in \overline{\mathcal{R}}$ for all $j \in \mathbb{N}$.*

Then T has a fixed point in X.

Proof. Let $x_0 \in X$ be such that $(x_0, Tx_0) \in \mathcal{R}$. Let $\{x_n\}$ be the Picard sequence generated by T with initial value x_0, then $(x_0, x_1) \in \mathcal{R}$. Therefore, we obtain from (1) and Lemma 1.8 that

$$\|x_1 - x_2, z\| = \|Tx_0 - Tx_1, z\| \preceq k\|x_0 - x_1, z\| \preceq \|x_0 - x_1, z\|.$$

Since T is $\mathcal{R}$-preserving with respect to $\|\cdot,\cdot\|$ we have $(x_1, x_2) = (Tx_0, Tx_1) \in \mathcal{R}$. Repetition of this process gives $(x_{n-1}, x_n) \in \mathcal{R}$ for all $n \in \mathbb{N}$. We shall show that the Picard sequence $\{x_n\}$ is a Cauchy sequence. Since $(x_{n-1}, x_n) \in \mathcal{R}$ for all $n \in \mathbb{N}$, therefore, for all $z \in X$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \|x_n - x_{n+1}, z\| &= \|Tx_{n-1} - Tx_n, z\| \\ &\preceq k\|x_{n-1} - x_n, z\|. \end{aligned}$$

Repetition of this process gives

$$\|x_n - x_{n+1}, z\| \preceq k^n\|x_0 - x_1, z\| \quad \text{for all} \quad z \in X. \tag{2}$$

If $m, n \in \mathbb{N}$ and $m > n$, then using (2) we obtain

$$\begin{aligned}
\|x_n - x_m, z\| &= \|x_n - x_{n+1} + x_{n+1} - x_m, z\| \\
&\preceq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_m, z\| \\
&\preceq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + \cdots + \|x_{m-1} - x_m, z\| \\
&\preceq k^n\|x_0 - x_1, z\| + k^{n+1}\|x_0 - x_1, z\| + \cdots + k^{m-1}\|x_0 - x_1, z\| \\
&= k^n \sum_{i=0}^{m-n-1} k^i \|x_0 - x_1, z\| \\
&\preceq k^n \sum_{i=0}^{\infty} k^i \|x_0 - x_1, z\| \\
&= k^n (e-k)^{-1}\|x_0 - x_1, z\|.
\end{aligned}$$

Since $\rho(k) < 1$ we have $\|k^n\| \to 0$ as $n \to \infty$, and so, by Lemma 1.3 $\{k^n\}$ is a c-sequence. Therefore, using Proposition 1.5 we obtain $\{k^n(e-k)^{-1}\|x_0 - x_1, z\|\}$ is a c-sequence. Hence, for every $c \in \operatorname{int}P$ there exists $n_0 \in \mathbb{N}$ such that $k^n(e-k)^{-1}\|x_0 - x_1, z\| \ll c$ for all $n > n_0$. Thus, it follows from the above inequality that $\{x_n\}$ is a Cauchy sequence. Since $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and X is $\mathcal{R}$-T-orbitally complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

We shall show that the point x^* is a fixed point of T. Suppose that the condition (a) of (iii) is satisfied. Then by continuity of T we have

$$x_{n+1} = Tx_n \to Tx^* \quad \text{as} \quad n \to \infty.$$

By uniqueness of limit we have $Tx^* = x^*$, i.e., x^* is a fixed point of T.

Suppose that the condition (b) of (iii) is satisfied. Then, there exists a subsequence $\{x_{n_j}\}$ such that $(x_{n_j}, x^*) \in \overline{\mathcal{R}}$ for all $n \in \mathbb{N}$. Therefore, using (1) and Remark 3.3 we obtain: for every $z \in X$

$$\begin{aligned}
\|x^* - Tx^*, z\| &= \|x^* - x_{n_j+1} + x_{n_j+1} - Tx^*, z\| \\
&\preceq \|x^* - x_{n_j+1}, z\| + \|Tx_{n_j} - Tx^*, z\| \\
&\preceq \|x^* - x_{n_j+1}, z\| + k\|x_{n_j} - x^*, z\|.
\end{aligned}$$

Since $x_n \to x^*$ as $n \to \infty$, by definition and Proposition 1.5 we have $\{\|x^* - x_{n_j+1}\|\}$ and $\{k\|x_{n_j} - x^*, z\|\}$ are c-sequences in P, and so, by Proposition 1.6 we have $\{\|x^* - x_{n_j+1}\| + k\|x_{n_j} - x^*, z\|\}$ is a c-sequence. Therefore, it follows from the above inequality that there exists $n_0 \in \mathbb{N}$ such that $\|x^* - Tx^*, z\| \ll c$ for all $j > n_0$, which yields with Lemma 1.3 that $\|x^* - Tx^*, z\| = \theta$ for all

$z \in X$. Therefore, by Lemma 2.4 we have $x^* - Tx^* = 0_X$, i.e., $Tx^* = x^*$. Thus, x^* is a fixed point of T. □

Example 3.5. Let $X = \mathbb{R}^3$, $A = \mathbb{R}^3$ with norm defined by $\|x\| = \max\{|x_1|, |x_2|, |x_3|\}$ and the product by $(x_1, x_2, x_3)\,(y_1, y_2, y_3) = (x_1y_1, x_2y_2, x_3y_3)$. Let $P = \{(x_1, x_2, x_3) \in A \colon x_1, x_2, x_3 \geq 0\}$ be the solid cone in A and define a mapping $\|\cdot, \cdot\| \colon X \times X \to P$ by

$$\|x, y\| = \big(|x_2y_3 - x_3y_2|, |x_1y_3 - x_3y_1|, |x_1y_2 - x_2y_1|\big)$$

for all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X$. Then, $(X, \|\cdot, \cdot\|)$ is a linear cone 2-normed space. Let C_1^z be the right circular cylinder with axis as z-axis and radius 1, i.e.,

$$C_1^z = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon x_1^2 + x_2^2 \leq 1\big\}$$

and define a mapping $T \colon X \to X$ by

$$T(x_1, x_2, x_3) = \begin{cases} \left(\dfrac{x_1}{4}, \dfrac{x_2}{4}, \dfrac{x_3}{4}\right), & \text{if } (x_1, x_2, x_3) \in C_1^z; \\ (\sin x_1, x_2, 0), & \text{otherwise.} \end{cases}$$

for all $(x_1, x_2, x_3) \in X$. Define a binary relation $\mathcal{R}$ on X by $\mathcal{R} = \{C_1^z \times C_1^z\} \cup P_y$, where $P_y = \{(x, y) \in X \times X \colon x_2 = y_2\}$ for $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$. Then it is easy to see that X is a $\mathcal{R}$-T-orbitally complete cone 2-normed space over Banach algebra A and T is $\mathcal{R}$-preserving with respect to $\|\cdot, \cdot\|$. Now one can see that T is a generalized $\mathcal{R}$-Lipschitz contraction with $k = (j, j, j)$, $\dfrac{1}{4} \leq j < 1$. Also, for any $(x_1, x_2, x_3) \in X$ such that $x_2 = 0$, $x_1 \leq 1$ and $x_3 \in \mathbb{R}$ we have $((x_1, x_2, x_3), T(x_1, x_2, x_3)) \in \mathcal{R}$. Note that, if $\{x_n\} \subset X$ is any Picard sequence such that $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then we must have $x = (0, 0, 0)$. Therefore, the condition (b) of (iii) of Theorem 3.4 is satisfied. Thus, all the conditions of Theorem 3.4 are satisfied and one can conclude the existence of fixed point of T by Theorem 3.4. Indeed, the set of all fixed point points of T is

$$\mathrm{Fix}(T) = \big\{(0, x_2, 0) \in X \colon x_2^2 > 1\big\} \cup \{(0, 0, 0)\}.$$

It is shown in the above example that the fixed point of mapping T satisfying the conditions of Theorem 3.4 may not be unique. In the next theorem, we give a sufficient condition for the uniqueness of fixed point of mapping T.

Theorem 3.6. *Suppose all the conditions of Theorem 3.4 are satisfied, then T has a fixed point in X. In addition, if for every $x^*, y^* \in \mathrm{Fix}(T)$ there exists $w \in X$ such that $(x^*, w), (y^*, w) \in \overline{\mathcal{R}}$, then T has a unique fixed point in X.*

Proof. The existence of fixed point $x^* \in X$ follows from Theorem 3.4. Suppose, x^* and y^* be two distinct fixed points of T, i.e., $Tx^* = x^* \neq y^* = Ty^*$. By assumption, there exists $w \in X$ such that $(x^*, w), (y^*, w) \in \overline{\mathcal{R}}$. Using Remark 3.3, Lemma 1.8 and (1) we obtain: for every $z \in X$

$$\|Tx^* - Tw, z\| \preceq k\|x^* - w, z\| \preceq \|x^* - w, z\| \text{ for all } z \in X.$$

Since T is $\mathcal{R}$-preserving with respect to $\|\cdot,\cdot\|$ we have $(Tx^*, Tw) = (x^*, Tw) \in \mathcal{R}$. Repetition of this process gives $(T^n x^*, T^n w) = (x^*, T^n w) \in \mathcal{R}$ for all $n \in \mathbb{N}$. Similarly, one can find $(T^n y^*, T^n w) = (y^*, T^n w) \in \mathcal{R}$ for all $n \in \mathbb{N}$. Therefore, using (1) we obtain: for every $z \in X$

$$\begin{aligned} \|x^* - y^*, z\| &= \|x^* - T^n w + T^n w - y^*, z\| \\ &\preceq \|T^n x^* - T^n w, z\| + \|T^n y^* - T^n w, z\| \\ &\preceq k\|T^{n-1} x^* - T^{n-1} w, z\| + k\|T^{n-1} y^* - T^{n-1} w, z\| \\ &= k\left[\|T^{n-1} x^* - T^{n-1} w, z\| + \|T^{n-1} y^* - T^{n-1} w, z\|\right]. \end{aligned}$$

Repetition of this process yeilds

$$\|x^* - y^*, z\| \preceq k^n \left[\|x^* - w, z\| + \|y^* - w, z\|\right]$$

for all $z \in X$ and for all $n \in \mathbb{N}$. Since $\rho(k) < 1$ we have $\|k^n\| \to 0$ as $n \to \infty$, and so, $\{k^n\}$ is a c-sequence, therefore, using Proposition 1.5 we obtain $k^n \left[\|x^* - w, z\| + \|y^* - w, z\|\right]$ is a c-sequence. Therefore, it follows from the above inequality that, for every $c \in \operatorname{int} P$ there exists $n_0 \in \mathbb{N}$ such that $\|x^* - y^*, z\| \ll c$ for all $n > n_0$, which yields with Lemma 1.3 that $\|x^* - y^*, z\| = \theta$ for all $z \in X$. Therefore, by Lemma 2.4 we have $x^* - y^* = 0_X$, i.e., $x^* = y^*$. This contradiction proves the uniqueness of the fixed point. □

Let $(X, \|\cdot,\cdot\|)$ be a linear cone 2-normed space over Banach algebra A, $\sqsubseteq$ be a partial order relation on X and $T\colon X \to X$ be a mapping. Then T is called $\sqsubseteq$-preserving with respect to $\|\cdot,\cdot\|$ if

$$x \sqsubseteq y, \|Tx - Ty, z\| \preceq \|x - y, z\| \; \forall z \in X \implies Tx \sqsubseteq Ty.$$

The space X is called $\sqsubseteq$-T-orbitally complete if every Cauchy sequence $\{x_n\} \subset O(T, x_0)$ such that $x_n \sqsubseteq x_{n+1}$ for all $n \in \mathbb{N}$ is convergent in X for all $x_0 \in X$.

The following corollary is an improved version of the results of Ran and Reurings [**22**] and Nieto and Rodríguez-López [**19**] in cone 2-normed space over Banach algebra A.

Corollary 3.7. *Let* $(X, \|\cdot,\cdot\|)$ *be a linear cone 2-normed space over Banach algebra* A, $\sqsubseteq$ *be a partial order relation on* X *and* $T\colon X \to X$ *be a mapping such that* X *is* $\sqsubseteq$-T*-orbitally complete. Suppose the following conditions are satisfied:*

(i) *there exists* $x_0 \in X$ *such that* $x_0 \sqsubseteq Tx_0$;

(ii) T *is* $\sqsubseteq$*-preserving with respect to* $\|\cdot,\cdot\|$;

(iii) *there exists* $k \in P$ *such that* $\rho(k) < 1$ *and*

$$\|Tx - Ty, z\| \preceq k\|x - y, z\| \text{ for all } x, y, z \in X \text{ with } x \sqsubseteq y;$$

(iv) *one of the following conditions is satisfied:*

(a) T *is continuous;*

(b) *if* $\{x_n\}$ *is the Picard sequence generated by* T *with initial value* x_0, $x_n \sqsubseteq x_{n+1}$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n_j}\}$ *such that* $x_{n_j} \sqsubseteq x$ *for all* $j \in \mathbb{N}$.

Then T *has a fixed point in* X.

Proof. Define the binary relation $\mathcal{R}$ on X by $\mathcal{R} = \{(x, y) : x \sqsubseteq y\}$. Now the proof follows from Theorem 3.4. □

Corollary 3.8 (Banach contraction principle in cone 2-normed space). *Let* $(X, \|\cdot,\cdot\|)$ *be a complete linear cone 2-normed space over Banach algebra* A *and* $T\colon X \to X$ *be a mapping. Suppose, there exists* $k \in P$ *such that* $\rho(k) < 1$ *and*

$$\|Tx - Ty, z\| \preceq k\|x - y, z\|$$

for all $x, y, z \in X$. *Then* T *has a unique fixed point in* X.

Proof. Define the binary relation $\mathcal{R}$ on X by $\mathcal{R} = X \times X$. Now the proof follows from Theorem 3.6. □

Remark 3.9. In Theorem 3.4, one can take the binary relation $\mathcal{R}$ as, e.g., the preorder (see, [**1**]), transitive relation, the tolerance relation associated with a partial order (see, [**1**]), strict order associated with a partial order; and can obtained the improved cone 2-normed version and variant of fixed point results considered in [**2**], [**26**], [**27**], [**28**] and [**11**].

The proof of the following theorem is similar to the proof of Theorem 3.4, therefore we omit it.

Theorem 3.10. *Let $(X, \|\cdot,\cdot\|)$ be a complete linear cone 2-normed space over Banach algebra A, $\mathcal{R}$ a binary relation on X, C be a nonempty subset of X and $T\colon C \to C$ be a generalized $\mathcal{R}$-Lipschitz mapping such that C is $\mathcal{R}$-T-orbitally closed. Suppose the following conditions are satisfied:*

(i) *there exists $x_0 \in C$ such that $(x_0, Tx_0) \in \mathcal{R}$;*

(ii) *T is $\mathcal{R}$-preserving with respect to $\|\cdot,\cdot\|$;*

(iii) *one of the following conditions is satisfied:*

  (a) *T is continuous;*

  (b) *if $\{x_n\}$ is the Picard sequence generated by T with initial value x_0, $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n_j}\}$ such that $(x_{n_j}, x) \in \overline{\mathcal{R}}$ for all $j \in \mathbb{N}$.*

Then T has a fixed point in C.

In the further discussion, we assume that the underlying space X is a real linear cone-2 normed space, that is, the space X taken over the real field.

In the next theorem, we show that the fixed point of a generalized $\mathcal{R}$-Lipschitz mapping on a $\mathcal{R}$-T-orbitally closed set can be approximated by Maan iteration scheme.

Theorem 3.11. *Let C be a nonempty convex subset of a complete cone 2-normed space X over Banach algebra A, $T\colon C \to C$ be a mapping such that $(1-k)C+kT(C) \subset C$ for all $0 < k < 1$ and C is $\mathcal{R}$-T-orbitally closed. Suppose, all the conditions of Theorem 3.10 are satisfied, then T has a fixed point $x^* \in C$. In addition, if $C \asymp Fix(T)$ and for arbitrary $y_0 \in C$ the Mann iteration scheme $\{y_n\}$ with initial value y_0 converges to the fixed point x^* with a rate estimated by*

$$\|y_n - x^*, z\| \preceq h^n L$$

for all $z \in X$, where $L, h \in P$ and $\rho(h) < 1$.

Proof. The existence of fixed point $x^* \in C$ follows from Theorem 3.10. Let $\{y_n\}$ be the Mann iteration scheme, i.e.,

$$y_{n+1} = (1 - \beta_n)y_n + \beta_n T y_n,$$

where $\beta_n \in (0,1]$ for all $n \in \mathbb{N}_0$, are such that $\sum_{i=0}^{\infty} \beta_i = \infty$. Since $C \asymp \mathrm{Fix}(T)$ we have $(y_n, x^*) \in \overline{\mathcal{R}}$ for all $n \in \mathbb{N}_0$. Therefore, for every $z \in X$ using Remark 3.3 we have

$$\begin{aligned}
\|y_{n+1} - x^*, z\| &= \|(1-\beta_n)y_n + \beta_n T y_n - x^*, z\| \\
&= \|(1-\beta_n)(y_n - x^*) + \beta_n(Ty_n - x^*), z\| \\
&\preceq (1-\beta_n)\|y_n - x^*, z\| + \beta_n\|Ty_n - x^*, z\| \\
&= (1-\beta_n)\|y_n - x^*, z\| + \beta_n\|Ty_n - Tx^*, z\| \\
&\preceq (1-\beta_n)\|y_n - x^*, z\| + \beta_n k\|y_n - x^*, z\| \\
&= [e - \beta_n(e-k)]\|y_n - x^*, z\| \\
&= h\|y_n - x^*, z\|
\end{aligned}$$

where $h = e - \beta_n(e-k)$. Repeating the above process we obtain

$$\|y_{n+1} - x^*, z\| \preceq h^{n+1}\|y_0 - x^*, z\| \quad \text{for all} \quad z \in X. \tag{3}$$

Since $(1-\beta_n)e$ and $\beta_n k$ commutes, using Lemma 1.7 we have

$$\begin{aligned}
\rho(h) &= \rho(e - \beta_n(e-k)) = \rho((1-\beta_n)e + \beta_n k) \\
&\leq \rho((1-\beta_n)e) + \rho(\beta_n k) \\
&= (1-\beta_n)\rho(e) + \beta_n\rho(k) \\
&= 1 - \beta_n + \beta_n\rho(k) \\
&= 1 - \beta_n(1-\rho(k)) \\
&< 1.
\end{aligned}$$

Let $L = \|y_0 - x^*, z\| \in P$ we obtain from (3) that

$$\|y_{n+1} - x^*, z\| \preceq h^{n+1}L \quad \text{for all} \quad z \in X. \tag{4}$$

Since $\rho(h) < 1$, we have $h^n \to \theta$ as $n \to \infty$, and so, by Lemma 1.3, for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $h^n L \ll c$ for all $n > n_0$. Therefore, we obtain from (4) that

$$\|y_n - x^*, z\| \ll c \quad \text{for all} \quad n > n_0, z \in X.$$

Thus, the Mann iteration scheme $\{y_n\}$ converges to x^* with a rate estimated by (4). □

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