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Three solutions for elliptic systems involving $p(x)$-biharmonic operators

Abstract. In this paper, we study the existence of solutions for elliptic systems with variable exponents. Under some suitable conditions and by applying an equivalent variational approach to a recent Ricceris three critical points theorem, we established the existence of at least three weak solutions.

Keywords. $p(x)$-biharmonic operator, variable exponent Sobolev space, critical point theorem.

Mathematics Subject Classification (2010): 35J35, 35J55, 47J30.

1 - Introduction

In this article, we consider the existence of solutions for the following system

$$\begin{aligned}
&\Delta(|\Delta u|^{p(x)-2}\Delta u) + |u|^{p(x)-2}u = \lambda F_u(x,u,v) + \mu G_u(x,u,v) \quad \text{in } \Omega,\\
&\Delta(|\Delta v|^{q(x)-2}\Delta v) + |v|^{q(x)-2}v = \lambda F_v(x,u,v) + \mu G_v(x,u,v) \quad \text{in } \Omega,\\
&\frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}\Big(|\Delta u|^{p(x)-2}\Delta u\Big) = \frac{\partial}{\partial \nu}\Big(|\Delta v|^{q(x)-2}\Delta v\Big) = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, with smooth boundary $\partial\Omega$, λ, μ are real parameters and p, $q \in C(\overline{\Omega})$ with $\frac{N}{2} < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x) < +\infty$, $\frac{N}{2} < q^- := \inf_{x\in\overline{\Omega}} q(x) \leq q^+ := \sup_{x\in\overline{\Omega}} q(x) < +\infty$. $F, G : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are functions such that $F(.,s,t), G(.,s,t)$ are measurable in Ω, for

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all $(s,t) \in \mathbb{R}^2$, $F(x,.,.)$ is C^1 in $\mathbb{R}^2$ for every $x \in \Omega$, F_u, F_v denote the partial derivatives of F with respect to u, v respectively.

The study of differential equations and variational problems with variable exponents has attracted intense research interests in recent years. Such problems arise from the study of electrorheological fluids, image processing, and the theory of nonlinear elasticity (see [**18**, **23**]).

There are many works devoted to the existence of solutions for variable exponent problems, both on bounded domain and unbounded domain, we refer to [**3**, **10**, **22**] as examples. For existence results on elliptic systems, we refer to [**2**, **11**, **19**, **21**].

The investigation of existence and multiplicity of solutions for problems involving biharmonic, p-biharmonic and $p(x)$-biharmonic operators has drawn the attention of many authors, see [**1**,**4**,**5**,**8**,**9**,**12**,**15**,**16**] and references therein. Candito and Livrea [**8**] considered the nonlinear elliptic Navier boundary-value problem

$$\begin{aligned} \Delta(|\Delta u|^{p-2}\Delta u) &= \lambda f(x,u) \quad \text{in} \quad \Omega, \\ u = \Delta u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2}$$

There the authors established the existence of infinitely many solutions. Here we point out that the $p(x)$-biharmonic operator possesses more complicated nonlinearities than p-biharmonic, for example, it is inhomogeneous and usually it does not have the so-called first eigenvalue, since the infimum of its principle eigenvalue is zero (see [**6**]).

We will use the notations such as h^-, h^+ and $h^{**}(x)$ where

$$h^- := \inf_{x\in\overline{\Omega}} h(x) \le h(x) \le h^+ := \sup_{x\in\overline{\Omega}} h(x) < +\infty,$$

$$h^{**}(x) = \begin{cases} \dfrac{Nh(x)}{N-2h(x)}, & \text{if } h(x) < \dfrac{N}{2}; \\ +\infty, & \text{if } h(x) \ge \dfrac{N}{2}. \end{cases}$$

Throughout this paper, we suppose the following assumptions.

There exist a positive constant C and two functions $\alpha, \beta \in C(\overline{\Omega})$ with $1 < \alpha^- \le \alpha^+$, $1 < \beta^- \le \beta^+$ and

$$\frac{1+\alpha^+}{p^-} + \frac{1+\beta^+}{q^-} < 1, \tag{3}$$

such that

(F_0) $|F_s(x,s,t)| \leq C|s|^{\alpha(x)}|t|^{\beta(x)+1}$, $\quad |F_t(x,s,t)| \leq C|s|^{\alpha(x)+1}|t|^{\beta(x)}$, for a.e. $x \in \Omega$ and all $(s,t) \in \mathbb{R}^2$.

(F_1) $F(x,s,t) < 0$, for all $x \in \Omega$ and $s,t \in]0,1[$.

(F_2) $F(x,s,t) \geq M > 0$, for all $x \in \Omega$, $s > s_0 > 1$ and $t > t_0 > 1$.

$G : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is function such that $G(.,s,t)$ is measurable in Ω for all $(s,t) \in \mathbb{R}^2$ and $G(x,.,.)$ is continuously differentiable in $\mathbb{R}^2$ for a.e. $x \in \Omega$. G_s, G_t are the partial derivatives of G which satisfy the following condition.

(G_0) $\sup_{\{|s|\leq\theta,|t|\leq\theta\}} \Big(|G_s(.,s,t)| + |G_t(.,s,t)|\Big) \in L^1(\Omega)$ for all $\theta > 0$.

Through this paper, we will consider the following spaces:

$$X_1 = \Big\{u \in W^{2,p(x)}(\Omega) : \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0\Big\} \text{ and } X_2 = \Big\{v \in W^{2,q(x)}(\Omega) : \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0\Big\}.$$

The goal of this paper is to prove the following result.

Theorem 1.1. *Assume (F_0), (F_1), (F_2) and (G_0) hold. Then there exist an open interval $\Lambda \subseteq (0,+\infty)$ and a positive real number r with the following property: for each $\lambda \in \Lambda$ there exists δ such that for each $\mu \in [-\delta,\delta]$, problem* (1) *has at least three weak solutions whose norms in $X_1 \times X_2$ are less than r.*

This article is organized as follows. In Section 2, we introduce the generalized Lebesgue-Sobolev spaces and some important related results. In Section 3, we use the general variational principle by B. Ricceri to prove the main result.

2 - Preliminaries

To study $p(x)$-Laplacian problems, we need some results on the spaces $L^{p(x)}(\Omega)$, $W^{k,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian used later.

Define the generalized Lebesgue space by

$$L^{p(x)}(\Omega) := \Big\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\Big\},$$

where $p \in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega}) := \big\{p \in C(\overline{\Omega}) : p(x) > 1 \quad \forall x \in \overline{\Omega}\big\}.$$

One introduces in $L^{p(x)}(\Omega)$ the norm

$$|u|_{p(x)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)} dx \leq 1\Big\}.$$

The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is a Banach space.

Proposition 2.1 (cf. [**14**]). *The space* $(L^{p(x)}(\Omega), |.|_{p(x)})$ *is separable, uniformly convex and its conjugate space is* $L^{q(x)}(\Omega)$ *where* $q(x)$ *is the conjugate function of* $p(x)$*, i.e*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \forall x \in \Omega.$$

For $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$ *we have*

$$\Big| \int_\Omega u(x)v(x)dx\Big| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)} |v|_{q(x)}.$$

The Sobolev space with variable exponents $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\right\},$$

where $D^\alpha u = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} u$ (the derivation in distributional sense) with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$, equipped with the norm

$$|u|_{k,p(x)} := \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)},$$

also becomes a Banach, separable and reflexive space. For more details, we refer the reader to [**13, 14**].

In this paper, we shall look for weak solutions of problem (1) in the space X defined by

$$X := X_1 \times X_2,$$

which is separable and reflexive Banach spaces with the norm

$$\|(u, v)\| = \|u\|_{p(x)} + \|v\|_{q(x)},$$

where $\|.\|_{p(x)}$ (resp. $\|.\|_{q(x)}$) is the norm of $W^{2,p(x)}(\Omega)$ (resp. $W^{2,q(x)}(\Omega)$),

$$\|u\|_{p(x)} = \inf\{\sigma > 0 : \int_\Omega \left(|\frac{\triangle u}{\sigma}|^{p(x)} + |\frac{\nabla u}{\sigma}|^{p(x)} + |\frac{u}{\sigma}|^{p(x)}\right) dx \leq 1\},$$

and

$$\|u\|_{q(x)} = \inf\{\sigma > 0 : \int_\Omega \left(|\frac{\triangle u}{\sigma}|^{q(x)} + |\frac{\nabla u}{\sigma}|^{q(x)} + |\frac{u}{\sigma}|^{q(x)}\right) dx \leq 1\}.$$

According to [**20**], the norm $|.|_{2,p(x)}$ is equivalent to the norm $|\triangle.|_{p(x)} + |.|_{p(x)}$ in the space $W^{2,p(x)}(\Omega)$. Consequently, the norms $|.|_{2,p(x)}$, $|\triangle.|_{p(x)} + |.|_{p(x)}$ and $\|.\|_{p(x)}$ are equivalent.

Proposition 2.2 (cf. [**4**]). *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) < p^{**}(x)$ for all $x \in \overline{\Omega}$, there is a continuous and compact embedding*

$$W^{2,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

Proposition 2.3 (cf. [**6**]). *If $2p(x) \geq N$ for all $x \in \overline{\Omega}$, then the set*

$$X_1 = \left\{ u \in W^{2,p(x)}(\Omega) : \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0 \right\}$$

is a closed subspace of $W^{2,p(x)}(\Omega)$.

Proposition 2.4. *The embedding $X \hookrightarrow C(\overline{\Omega}) \times C(\overline{\Omega})$ is compact whenever $p^- > \frac{N}{2}$ and $q^- > \frac{N}{2}$. So there is a constant $C_0 > 0$ such that*

$$C_0 := \max \left\{ \sup_{u \in W^{2,p(x)}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u(x)|}{\|u\|_{p(x)}}, \sup_{v \in W^{2,q(x)}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |v(x)|}{\|v\|_{q(x)}} \right\} < \infty. \quad (4)$$

Proof. It is well known that $W^{2,p(x)}(\Omega) \times W^{2,q(x)}(\Omega) \hookrightarrow W^{2,p^-}(\Omega) \times W^{2,q^-}(\Omega)$ are all continuous embedding. And the embedding $W^{2,p^-}(\Omega) \times W^{2,q^-}(\Omega) \hookrightarrow C(\overline{\Omega}) \times C(\overline{\Omega})$ is compact when $p^- > \frac{N}{2}$ and $q^- > \frac{N}{2}$. So we get, the embedding $X \hookrightarrow C(\overline{\Omega}) \times C(\overline{\Omega})$ is compact when $p^- > \frac{N}{2}$ and $q^- > \frac{N}{2}$. $\square$

Using the similar proof method with [**13**], we have the following result.

Proposition 2.5. *Let $I_{p(x)}(u) = \int_\Omega | \triangle u |^{p(x)} + |u|^{p(x)} dx$, for $u \in W^{2,p(x)}(\Omega)$ we have*

1. *For $u \neq 0$, $\|u\|_{p(x)} = \beta \Leftrightarrow I(\frac{u}{\beta}) = 1$;*
2. $\|u\|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow I(u) < 1 (= 1, > 1)$;
3. $\|u\|_{p(x)} \leq 1 \Rightarrow \|u\|^{p^+}_{p(x)} \leq I(u) \leq \|u\|^{p^-}_{p(x)}$;
4. $\|u\|_{p(x)} \geq 1 \Rightarrow \|u\|^{p^-}_{p(x)} \leq I(u) \leq \|u\|^{p^+}_{p(x)}$;
5. $\lim_{k\to+\infty} \|u_k\|_{p(x)} = 0 \Leftrightarrow \lim_{k\to+\infty} I(u_k) = 0$;

6. $\lim_{k\to+\infty} \|u_k\|_{p(x)} = +\infty \Leftrightarrow \lim_{k\to+\infty} I(u_k) = +\infty$.

To prove our main result, we will use the following result proved in [**17**] that, on the basis of [**7**], can be equivalently stated as follows:

Proposition 2.6. *Let X be a reflexive real Banach space, $\Phi\colon X \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* and Φ is bounded on each bounded subset of X; $\Psi\colon X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Moreover, assume that*

$$\lim_{\|u\|_X\to+\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty, \tag{5}$$

for all $\lambda \in (0, +\infty)$, and that there exist $\rho \in \mathbb{R}$ and $u_0, u_1 \in X$ such that

$$\Phi(u_0) < \rho < \Phi(u_1), \tag{6}$$

$$\inf_{u\in\Phi^{-1}((-\infty,\rho])} \Psi(u) > \frac{(\Phi(u_1) - \rho)\Psi(u_0) + (\rho - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}. \tag{7}$$

Then, there exist an open interval $\Lambda \subseteq (0, +\infty)$ and a positive real number r with the following property:

For every $\lambda \in \Lambda$ and every C^1 functional $J\colon X \mapsto \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the equation

$$\Phi'(u) + \lambda\Psi'(u) + \mu J'(u) = 0,$$

has at least three solutions in X whose norms are less than r.

For each $u \in X_1$, we define

$$T(u) = \int_\Omega \frac{1}{p(x)} \left(|\triangle u|^{p(x)} + |u|^{p(x)}\right) dx.$$

Then, the operator $L := T' : X_1 \to X_1^*$, where X_1^* is the dual space of X_1, defined by

$$\langle L(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v + |u^{p(x)-2}uvdx \quad \forall v \in X_1, \tag{8}$$

satisfies the assertions of the following proposition.

Proposition 2.7 (cf. [**12**]).

1. *L is continuous, bounded and strictly monotone.*
2. *L is of (S_+) type.*
3. *L is a homeomorphism.*

3 - Proof of the main result

Definition 3.1. We say that $(u,v) \in X$ is a weak solution of problem (1) if

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta\varphi + |u|^{p(x)-2}u\varphi\, dx + \int_\Omega |\Delta v|^{q(x)-2}\Delta v\Delta\psi + |v|^{q(x)-2}v\psi\, dx$$
$$-\lambda\int_\Omega F_u(x,u,v)\varphi\, dx - \lambda\int_\Omega F_v(x,u,v)\psi\, dx - \mu\int_\Omega G_u(x,u,v)\varphi\, dx$$
$$-\mu\int_\Omega G_v(x,u,v)\psi\, dx = 0,$$

for all $(\varphi,\psi) \in X$.

Define the functional $E_{\lambda,\mu} : X \to \mathbb{R}$, by

$$E_{\lambda,\mu}(u,v) = \Phi(u,v) + \lambda\Psi(u,v) + \mu J(u,v),$$

for all $(u,v) \in X$, where

$$\Phi(u,v) = \int_\Omega \frac{1}{p(x)}\left(|\triangle u|^{p(x)} + |u|^{p(x)}\right) dx + \int_\Omega \frac{1}{q(x)}\left(|\triangle v|^{q(x)} + |v|^{q(x)}\right) dx,$$

$$\Psi(u,v) = -\int_\Omega F(x,u,v)dx, \qquad J(u,v) = -\int_\Omega G(x,u,v)dx.$$

The functionals $\Phi, \Psi, J : X \to \mathbb{R}$ are well defined, Gâteaux differentiable functionals whose Gâteaux derivatives at $(u,v) \in X$ are given by

$$\langle \Phi'(u,v),(\varphi,\psi)\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta\varphi + |u|^{p(x)-2}\, u\varphi dx$$
$$+\int_\Omega |\Delta v|^{q(x)-2}\Delta v\Delta\psi + |v|^{q(x)-2}\, v\psi dx,$$

$$\langle \Psi'(u,v),(\varphi,\psi)\rangle = -\int_\Omega F_u(x,u,v)\varphi dx - \int_\Omega F_v(x,u,v)\psi dx,$$
$$\langle J'(u,v),(\varphi,\psi)\rangle = -\int_\Omega G_u(x,u,v)\varphi dx - \int_\Omega G_v(x,u,v)\psi dx,$$

for all $(\varphi, \psi) \in X$.

Hence, $(u, v) \in X$ is a weak solution of (1) if and only if (u, v) is a critical point of the functional $E_{\lambda,\mu}$.

We now turn to the proof of Theorem 1.1. First, we check the conditions of Proposition 2.6.

According to Proposition 2.7, of course, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, moreover, Ψ and J are continuously Gâteaux differentiable functional whose Gâteaux derivative are compact. Obviously, Φ is bounded on each bounded subset of X under our assumptions.

Moreover, we have

$$\lim_{\|(u,v)\|\to\infty} (\Phi(u, v) + \lambda\Psi(u, v)) = +\infty,$$

for all $\lambda \in (0, +\infty)$. Indeed,

$$\begin{aligned} \Phi(u, v) &= \int_\Omega \frac{1}{p(x)} \left(|\triangle u|^{p(x)} + |u|^{p(x)}\right) dx + \int_\Omega \frac{1}{q(x)} \left(|\triangle v|^{q(x)} + |v|^{q(x)}\right) dx \\ &\geq \frac{1}{p^+} \min\left(\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\right) + \frac{1}{q^+} \min\left(\|v\|_{q(x)}^{q^-}, \|v\|_{q(x)}^{q^+}\right). \end{aligned} \tag{9}$$

By (F_0), we have $|F(x, u, v)| \leq C|u|^{\alpha(x)+1}|v|^{\beta(x)+1}$, for all $(u, v) \in X$.

Therefore

$$\begin{aligned} \Psi(u, v) &= -\int_\Omega F(x, u, v)dx \\ &\geq -C \int_\Omega |u|^{\alpha(x)+1}|v|^{\beta(x)+1} dx \\ &\geq -C|\Omega| \max\left(\|u\|_\infty^{1+\alpha^+}, \|u\|_\infty^{1+\alpha^-}\right) \max\left(\|v\|_\infty^{1+\beta^+}, \|v\|_\infty^{1+\beta^-}\right). \end{aligned}$$

We know that $W^{2,p(x)}(\Omega)$ and $W^{2,q(x)}(\Omega)$ are continuously embedded in $C(\overline{\Omega})$, so there is a constant C_0 such that

$$\Psi(u, v) \geq -CC_0|\Omega| \max\left(\|u\|_{p(x)}^{1+\alpha^+}, \|u\|_{p(x)}^{1+\alpha^-}\right) \max\left(\|v\|_{q(x)}^{1+\beta^+}, \|v\|_{q(x)}^{1+\beta^-}\right). \tag{10}$$

Without loss of generality, we will distinguish two cases since $\|(u, v)\|$ is considered to tend to $+\infty$. We will restrict ourselves to the cases when $\|u\|_{p(x)} \to +\infty$,

$\|v\|_{q(x)}$ bounded and when $\|u\|_{p(x)}, \|v\|_{q(x)} \to +\infty$. In view of (3), there exist $p_1 \in (1, p^-)$ and $q_1 \in (1, q^-)$ such that $\dfrac{1+\alpha^+}{p_1} + \dfrac{1+\beta^+}{q_1} = 1$. Hence from (10) and Young's inequality we get

$$\Psi(u,v) \geq -CC_0|\Omega| \left(\frac{1+\alpha^+}{p_1} \|u\|_{p(x)}^{p_1} + \frac{1+\beta^+}{q_1} \max\left(\|v\|_{q(x)}^{q_1}, 1 \right) \right).$$

The above inequality and (9) imply the coercivity of $\Phi + \lambda\Psi$ since $p_1 < p^-$. On the other hand, let $\|u\|_{p(x)}, \|v\|_{q(x)} \to +\infty$, we have

$$\begin{aligned} \Psi(u,v) &\geq -C|\Omega| \|u\|_\infty^{1+\alpha^+} \|v\|_\infty^{1+\beta^+} \\ &\geq -CC_0|\Omega| \left(\frac{1+\alpha^+}{p_1} \|u\|_{p(x)}^{p_1} + \frac{1+\beta^+}{q_1} \|v\|_{q(x)}^{q_1} \right). \end{aligned}$$

Then, the coercivity of $\Phi+\lambda\Psi$ can be easily deduced since $p_1 < p^-$ and $q_1 < q^-$. Hence (5) of Proposition 2.6 is verified.Now, from (F_1) we can choose $\delta > 1$ such that $F(x,s,t) > 0$, for all $s,t > \delta$, $x \in \Omega$.

Then using (F_2) we get $F(x,s,t) \geq 0 = F(x,0,0) \geq F(x,\sigma_1,\sigma_2)$, $\forall s,t > \delta$, $\sigma_1, \sigma_2 \in (0,1)$.

Let a, b be two real numbers such that $0 < a < \min\{1, C_0\}$, with C_0 given by (4) and $b > \delta$ such thatmin$\left\{ b^{p^-}, b^{q^-} \right\} |\Omega| > 1$. Then we obtain

$$\int_\Omega \sup_{0<|u|,|v|<a} F(x,u,v)dx \leq 0 < \min\left\{ \frac{a^{p^+}}{C_0^{p^+} b^{p^-}}, \frac{a^{q^+}}{C_0^{p^+} b^{q^-}} \right\} \int_\Omega F(x,b,b)dx.$$

Set

$$\rho := \min\left\{ \frac{1}{p^+} \left(\frac{a}{C_0} \right)^{p^+}, \frac{1}{q^+} \left(\frac{a}{C_0} \right)^{q^+} \right\}.$$

By choosing $(u_0, v_0) = (0,0)$ and $(u_1, v_1) = (b,b)$, we obtain

$$\Phi(u_0,v_0) = \Psi(u_0,v_0) = 0,$$

$$\Phi(u_1,v_1) = \int_\Omega \frac{1}{p(x)} b^{p(x)} + \frac{1}{q(x)} b^{q(x)} dx \geq \left(\frac{1}{p^+} b^{p^-} + \frac{1}{q^+} b^{q^-} \right) |\Omega| > \rho.$$

Hence

$$\Phi(u_0,v_0) < \rho < \Phi(u_1,v_1).$$

Therefore (6) of Proposition 2.6 is verified.

On the other hand, we have

$$-\frac{(\Phi((u_1,v_1)-\rho)\Psi(u_0,v_0)+(\rho-\Phi(u_0,v_0))\Psi(u_1,v_1)}{\Phi(u_1,v_1)-\Phi(u_0,v_0)}=-\rho\frac{\Psi(u_1,v_1)}{\Phi(u_1,v_1)}$$

$$=\rho\frac{\int_\Omega F(x,b,b)dx}{\int_\Omega \frac{1}{p(x)}b^{p(x)}+\frac{1}{q(x)}b^{q(x)}dx}>0.$$

Let $(u,v)\in X$ be such that $\Phi(u,v)\leq\rho$. Thus

$$\Phi(u,v)\geq\frac{1}{p^+}I_{p(x)}(u)+\frac{1}{q^+}I_{q(x)}(v),$$

which implies that

$$I_{p(x)}(u)\leq p^+\rho<1 \quad \text{and} \quad I_{q(x)}(v)\leq q^+\rho<1.$$

According to Proposition 2.5, we get

$$\|u\|_{p(x)}\leq 1 \quad \text{and} \quad \|v\|_{q(x)}\leq 1.$$

Therefore

$$\frac{1}{p^+}\|u\|_{p(x)}^{p^+}+\frac{1}{q^+}\|u\|_{q(x)}^{q^+}\leq\Phi(u,v)\leq\rho.$$

Taking into account that

$$|u(x)|\leq C_0\left(p^+\rho\right)^{\frac{1}{p^+}}<a \quad \text{and} \quad |v(x)|\leq C_0\left(q^+\rho\right)^{\frac{1}{q^+}}<a,$$

for all $x\in\Omega$ and $(u,v)\in X$.

It follows

$$-\inf_{(u,v)\in\Phi^{-1}(-\infty,\rho]}\Psi(u,v)=\sup_{u\in\Phi^{-1}(-\infty,\rho]}-\Psi(u,v)$$
$$\leq\int_\Omega\sup_{0<|u|,|v|<a}F(x,u,v)dx\leq 0.$$

Then

$$\inf_{(u,v)\in\Phi^{-1}(-\infty,\rho]}\Psi(u,v)>\frac{(\Phi(u_1,v_1)-r)\Psi(u_0,v_0)+(\rho-\Phi(u_0,v_0))\Psi(u_1,v_1)}{\Phi(u_1,v_1)-\Phi(u_0,v_0)}.$$

Which means that condition (7) in Proposition 2.6 is obtained.

Hence, in view of Proposition 2.6, the proof of Theorem 1.1 is achieved.

In the next, we consider the following system

(11)
$$\Delta(|\Delta u|^{p(x)-2}\Delta u)+|u|^{p(x)-2}u=\lambda\Big(|u|^{\alpha(x)-1}u|v|^{\alpha(x)+1}-uv^2\Big)+\mu|u|^{\beta(x)-1}u \ \text{ in } \Omega,$$
$$\Delta(|\Delta v|^{q(x)-2}\Delta v)+|v|^{q(x)-2}v=\lambda\Big(|u|^{\alpha(x)+1}|v|^{\alpha(x)-1}v-u^2v\Big)+\mu|v|^{\gamma(x)-1}v \ \text{ in } \Omega,$$
$$\frac{\partial u}{\partial \nu}=\frac{\partial}{\partial \nu}\Big(|\Delta u|^{p(x)-2}\Delta u\Big)=\frac{\partial}{\partial \nu}\Big(|\Delta v|^{q(x)-2}\Delta v\Big)=\frac{\partial v}{\partial \nu}=0 \ \text{ on } \partial\Omega,$$

where

$$\beta,\gamma\in C_+(\overline{\Omega}),\quad \beta^+<p^- \quad \text{and} \quad \gamma^+<q^-, \tag{12}$$

and α satisfies

$$(\alpha^+ +1)\left(\frac{1}{p^-}+\frac{1}{q^-}\right)<1. \tag{13}$$

Corollary 3.1. *Let $\alpha\in C_+(\overline{\Omega})$ satisfying* (13). *Then there exist an open interval K of $(0,+\infty)$ and a positive real number r such that, for every $\lambda\in K$ and for two functions β,γ satisfying* (12), *there exists $\delta>0$ such that for each $\mu\in[-\delta,\delta]$ the system* (11) *has at least three solutions whose norms in X are less than r.*

Proof. Let the functional Φ defined as before and set

$$\Psi(u,v)=-\int_\Omega\left(\frac{1}{\alpha(x)+1}|u|^{\alpha(x)+1}|v|^{\alpha(x)+1}-\frac{1}{2}u^2v^2\right)dx,$$

and

$$J(u,v)=-\int_\Omega\left(\frac{1}{\beta(x)}|u|^{\beta(x)}+\frac{1}{\gamma(x)}|v|^{\gamma(x)}\right)dx.$$

Clearly Ψ' and J' are compact. From Proposition 2.4, there exists $C_0>0$ such that

$$\begin{aligned}\Psi(u,v)&\geq-\int_\Omega\left(\frac{1}{\alpha(x)+1}\|u\|_\infty^{\alpha(x)+1}\|v\|_\infty^{\alpha(x)+1}\right)dx\\&\geq-\frac{|\Omega|}{\alpha^-+1}\max\left(\|u\|_\infty^{1+\alpha^+},\|u\|_\infty^{1+\alpha^-}\right)\max\left(\|v\|_\infty^{1+\alpha^+},\|v\|_\infty^{1+\alpha^-}\right)\\&\geq-\frac{C_0|\Omega|}{\alpha^-+1}\max\left(\|u\|_{p(x)}^{1+\alpha^+},\|u\|_{p(x)}^{1+\alpha^-}\right)\max\left(\|v\|_{q(x)}^{1+\alpha^+},\|v\|_{q(x)}^{1+\alpha^-}\right).\end{aligned}$$

As in proof of Theorem 1.1, we may assume that $\|u\|_{p(x)}, \|v\|_{q(x)} \to +\infty$. Then

$$\Psi(u,v) \geq -\frac{C_0|\Omega|}{\alpha^- + 1} \|u\|_{p(x)}^{1+\alpha^+} \|v\|_{q(x)}^{1+\alpha^+}.$$

Since $(\alpha^+ + 1)\left(\frac{1}{p^-} + \frac{1}{q^-}\right) < 1$, there exist $p_2 \in (1, p^-)$ and $q_2 \in (1, q^-)$ such that

$$(\alpha^+ + 1)\left(\frac{1}{p^-} + \frac{1}{q^-}\right) = 1.$$

In view of Young's inequality, we obtain

$$\Psi(u,v) \geq -\frac{C_0|\Omega|}{\alpha^- + 1}\left(\frac{\alpha^+ + 1}{p_2}\|u\|_{p(x)} + \frac{\alpha^+ + 1}{q_2}\|v\|_{q(x)}\right).$$

This and (9) imply that

$$\lim_{\|(u,v)\| \to \infty} (\Phi(u,v) + \lambda \Psi(u,v)) = +\infty,$$

for all $\lambda \in (0, +\infty)$.

Let

$$H(x,s,t) = \frac{1}{\alpha(x)+1}|s|^{\alpha(x)+1}|t|^{\alpha(x)+1} - \frac{1}{2}s^2t^2.$$

Since $\alpha(x) > 1$ for all $x \in \Omega$, we can choose $\delta > 1$ such that

$$H(x,s,t) > 0 = H(x,0,0) \geq H(x,\sigma_1,\sigma_2), \quad \forall s,t > \delta, \quad \sigma_1, \sigma_2 \in (0,1).$$

Then adapting the same technique as in the proof of the previous theorem, we deduce that all the assumptions of Proposition 2.6 hold. Hence, our conclusion follows from Proposition 2.6. □

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