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Low genus curves with low degree on a general quintic 3-fold are finite and they have maximal rank

Abstract. Let $W \subset \mathbb{P}^4$ be a general quintic 3-fold. Fix integers d, g with $1 \le g \le 3$ and $g+3 \le d \le 11$. In this paper we prove that W contains only finitely many smooth curves $C \subset \mathbb{P}^4$ of degree d and genus g, all of them smooth and isolated points of the Hilbert scheme of W and that each such C has maximal rank, i.e. $h^1(\mathcal{I}_C(t)) \cdot h^0(\mathcal{I}_C(t)) = 0$ for all $t \in \mathbb{N}$.

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Clemens conjectured the finiteness of the set of rational curves of prescribed degree on a general quintic hypersurface of $\mathbb{P}^4$ ([**1**], [4], [**5**], [**6**], [**13**], [**14**], [**15**], [**16**], [**17**], [**21**], [**22**]; some authors also considered curves with positive genus).

Let $M_{d,g}$ denote the set of all smooth curves $C \subset \mathbb{P}^4$ with degree d, genus g and $h^1(\mathcal{O}_C(1)) = 0$. Let $X \subset \mathbb{P}^r$, $r \ge 2$, be any integral curve. Recall that X is said to have *maximal rank* (or *maximal rank in* $\mathbb{P}^r$) if for all integers $t > 0$ either $h^0(\mathbb{P}^r, \mathcal{I}_{X,\mathbb{P}^r}(t)) = 0$ or $h^1(\mathbb{P}^r, \mathcal{I}_{X,\mathbb{P}^r}(t)) = 0$. We always specify the ambient projective space ($\mathbb{P}^4$ for Theorem 1) for the following reason. If X is contained in a proper linear subspace of $\mathbb{P}^r$ (call N the linear span of X), then X has maximal rank if and only if X is a projectively normal curve of N, because $h^1(N, \mathcal{I}_{X,N}(t)) = h^1(\mathbb{P}^r, \mathcal{I}_{X,\mathbb{P}^r}(t))$ for all $t \in \mathbb{Z}$ and $h^0(\mathbb{P}^r, \mathcal{I}_{X,\mathbb{P}^r}(t)) > 0$ for all $t > 0$.

For any curve C contained in a smooth variety W let $N_{C,W}$ denote the normal bundle of C in W. The vector space $H^0(N_{C,W})$ is the tangent space at $[C]$ of the Hilbert scheme of W, while $H^1(N_{C,W})$ is an obstruction space for

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the same functor, so that if $H^1(N_{C,W}) = 0$, then the Hilbert space of W at $[C]$ is smooth and of dimension $h^0(N_X, W)$ at $[C]$. If $C \in M_{d,g}$ and $W \subset \mathbb{P}^4$ is a smooth quintic 3-fold, then $\chi(N_{C,W}) = 0$. Hence $h^i(N_{C,W}) = 0$, $i = 0, 1$, if and only if C is a smooth and isolated point of the Hilbert scheme of W.

Theorem 1. *Let W be a general quintic hypersurface of $\mathbb{P}^4$. Then W contains only finitely many elements $C \in M_{d,g}$, $1 \le g \le 3$, $g+3 \le d \le 11$, all of them have maximal rank, and they are smooth isolated points of the Hilbert scheme of W, i.e. $h^i(N_{C,W}) = 0$, $i = 0, 1$; they are non-degenerate if and only if $d \ne g + 3$.*

When $1 \le g \le 3$ and $d < g+3$, then we come into two cases: $(d, g) = (3, 1)$, i.e. plane cubics, and $(d, g) = (4, 3)$, i.e. plane quartics, i.e. canonically embedded non-hyperelliptic genus 3 curves. A general quintic hypersurface W has 2875 irreducible families of plane quartics, each of them parametrized by an open subset of a projective plane (they are of the form $N \cap W = L \cup C$ with N a plane containing one of the 2875 lines $L \subset W$), and 609,250 plane cubics (they are of the form $N \cap W = D \cup C$ with N a plane spanned by one of the 609,250 conics $D \subset W$) ([**16**, Theorem 3.1], [**1**, Remark 4]).

Take $C \in M_{d,g}$ with maximal rank. Since $5d + 1 - g < \binom{9}{4}$, we have $h^1(\mathcal{I}_C(5)) = 0$. Now take a general quintic $W \subset \mathbb{P}^5$ and fix any $C \in M_{d,g}$ with $C \subset W$. To prove Theorem 1 we need to prove that for each positive integer t either $h^1(\mathcal{I}_C(t)) = 0$ or $h^0(\mathcal{I}_C(t)) = 0$. In particular we need to prove that $h^1(\mathcal{I}_C(5)) = 0$. To prove that any $C \subset W$ satisfies $h^1(\mathcal{I}_C(5)) = 0$ is the key part of the proof. Finiteness, maximal rank and $h^i(N_{C,W}) = 0$, $i = 0, 1$, will easily follow after we prove that each $C \subset W$ satisfies $h^1(\mathcal{I}_C(5)) = 0$. A key part of the proof that $h^i(N_{C,W}) = 0$, $i = 0, 1$, for all C contained in a general quintic hypersuface $W \subset \mathbb{P}^4$ is [**17**, Theorem 1.2], which says that this is true for at least one $C \subset W$. L. Knutsen proved the existence of curves C with $h^i(N_{C,W}) = 0$, $i = 0, 1$, for other Calabi-Yau 3-folds for certain degrees and genera ([**17**], [**18**]).

The case $g = 0$ is also true, but a stronger result (true also for some singular rational curves) was proved by E. Cotterill ([**5**, Theorems 1.1 and 1.3]).

It should be difficult to prove the second part of a result like Theorem 1 for certain (d, g) without proving first that $h^1(\mathcal{I}_C(5)) = 0$, e.g. using the full incidence correspondence $\mathbb{I}_{d,g}$ instead of the partial incidence varieties $\mathbb{I}_{d,g;0}$ or $\mathbb{I}_{d,g;1}$ introduced in section 1, because the full incidence correspondence could be reducible (it is reducible if $d \ge 12$ and $g = 0$ by [**14**, Proposition 3.2]). Restricting the incidence correspondence was also effectively used in [**13**], [**14**] and [**5**], while T. Johnsen and S. Kleiman stressed that maximal rank is a simple consequence of finiteness and to have (after restricting the data) an irreducible incidence correspondence ([**14**, page 132]).

1 - Preliminaries

Let $\mathcal{W}$ denote the set of all smooth quintic hypersurfaces $W \subset \mathbb{P}^4$. Let $\mathcal{W}_1$ be the set of all $W \in \mathcal{W}$ containing only finitely many lines L, each of them with $N_{L,W} \cong \mathcal{O}_L(-1)^{\oplus 2}$ and pairwise disjoint (i.e. containing no reducible conic), only finitely many conics, no reducible curve with rational components and degree at most 4 and only finitely many degree 4 rational curves, all of them spanning $\mathbb{P}^4$. $\mathcal{W}_1$ is a non-empty open subset of $\mathcal{W}$ by [**16**] and [**5**, Theorems 1.1 and 1.3].

For all d, g with $1 \le g \le 3$ and $d \ge g+3$ let $M_{d,g}$ denote the set of all smooth curves $C \subset \mathbb{P}^4$ with degree d and genus g. Let $M'_{d,g}$ be the set of all $C \in M_{d,g}$ contained in a hyperplane of $\mathbb{P}^4$. With our assumptions on d and g every $C \in M_{d,g}$ satisfies $h^1(\mathcal{O}_C(1)) = 0$ and it is not contained in a plane. The Euler's sequence of $T\mathbb{P}^4$ shows that $T\mathbb{P}^4$ is a quotient of $\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5}$. Hence N_C is a quotient of $\mathcal{O}_C(1)^{\oplus 5}$. Since C is a curve, we have $h^2(\mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on C. Since $h^1(\mathcal{O}_C(1)) = 0$, we get that $M_{d,g}$ is smooth of dimension $5d+1-g$ and that for every hyperplane $H \subset \mathbb{P}^4$ the scheme $M_{d,g}(H)$ is smooth, and $\dim M_{d,g}(H) = 4d$. Since $h^1(\mathcal{O}_C(1)) = 0$ for all $C \in M_{d,g}$ (resp. $C \in M_{d,g}(H)$) we easily see that $M_{d,g}$ and $M_{d,g}(H)$ are irreducible. We write $\mathcal{I}_C$ for the ideal sheaf of C in $\mathbb{P}^4$ and write $H^i(\mathcal{I}_C(t))$ and $h^i(\mathcal{I}_C(t)) := \dim H^i(\mathcal{I}_C(t))$ instead of $H^i(\mathbb{P}^4, \mathcal{I}_C(t))$ and $h^i(\mathbb{P}^4, \mathcal{I}_C(t))$.

Remark 1. Fix $C \in M_{d,g}$, $g > 0$. If $C \notin M'_{d,g}$ (resp. $C \in M'_{d,g}$), then $h^1(\mathcal{I}_C(5)) = 0$ if $d \le 9$ (resp. $d \le 8$) by [**9**, Part (ii) of Theorem at page 492], because $g > 0$.

We need the following lemma, which is a variation of [**4**, Lemma 2]; it is a consequence of the bilinear lemma and it was used several times in the proof of [**6**, Theorem 4.1].

Lemma 1. *Fix integer $t \ge 2$, $r \ge 3$ and an integral and non-degenerate curve $T \subset \mathbb{P}^r$ such that $h^1(\mathcal{I}_T(t)) > 0$. Let $V \subseteq H^0(\mathcal{O}_{\mathbb{P}^r}(1))$ be a linear subspace such that $h^1(M, \mathcal{I}_{M\cap T,M}(t)) = 0$ for every hyperplane $M \subset \mathbb{P}^r$ whose equation is in $V \setminus \{0\}$. Then $h^1(\mathcal{I}_T(t-1)) \ge \dim(V) - 1 + h^1(\mathcal{I}_T(t))$.*

1.1 - *Reduction to the proof that $h^1(\mathcal{I}_C(5)) = 0$*

In the next two sections we will prove that every $C \in M_{d,g}$ contained in a general quintic hypersurface $W \subset \mathbb{P}^4$ satisfies $h^1(\mathcal{I}_C(5)) = 0$. Assume for the moment to know this statement. Let $\mathcal{U}$ be the set of all $W \in \mathcal{W}$ such that for all $1 \le g \le 3$ and $d \ge g+3$ every $C \in M_{d,g}$ contained in W satisfies $h^1(\mathcal{I}_C(5)) = 0$. By assumption $\mathcal{U}$ contains a non-empty open subset of $|\mathcal{O}_{\mathbb{P}^4}(5)|$.

Fix integers d, g with $1 \le g \le 3$ and $g+3 \le d \le 11$. Since $5d+1-g < \binom{8}{4}$, a general $C \in M_{d,g}$ has $h^1(\mathcal{I}_C(5)) = 0$ (for the case $d \ge g+4$ see [**3**], for the case $d = g+3$ and hence $C \in M'_{d,g}$ see [**2**]). Let $U_{d,g}$ denote the set of all $C \in M_{d,g}$ with $h^1(\mathcal{I}_C(5)) = 0$. $U_{d,g}$ is a smooth and irreducible and it has dimension $5d+1-g$. Let $\mathbb{I}_{d,g;0}$ (resp. $\mathbb{I}_{d,g;1}$) be the set of all pairs (C, W) with $W \in \mathcal{U}$ (resp. $W \in |\mathcal{O}_{\mathbb{P}^4}(5)|$), $C \in U_{d,g}$ and $C \subset W$. Let $\pi_2 : \mathbb{I}_{d,g;1} \to |\mathcal{O}_{\mathbb{P}^4}(5)|$ and $\pi_1 : \mathbb{I}_{d,g;1} \to U_{d,g}$ denote the projections. Since each fiber of π_1 is a projective space of dimension $\binom{9}{4} - 2 - 5d + g$, $\mathbb{I}_{d,g;1}$ is irreducible. Since $\mathcal{U}$ contains a non-empty open subset of $|\mathcal{O}_{\mathbb{P}^5}(5)|$ and π_2 is dominant, $\mathbb{I}_{d,g;0} = \pi_2^{-1}(\mathcal{U})$ is irreducible. A dimensional count gives that a general $W \in \mathcal{U}$ contains only finitely many elements of $M_{d,g}$. Fix a general $W \in \mathcal{U}$. Since $\mathbb{I}_{d,g;0}$ is irreducible, to prove that all $C \in M_{d,g}$ contained in W satisfies $h^i(N_{C,W}) = 0$, $i = 0, 1$, it is sufficient to know that W contains one $C \in M_{d,g}$ with $h^i(N_{C,W}) = 0$, $i = 0, 1$, which is the result proved in [**17**, Theorem 1.2]. C has also maximal rank in $\mathbb{P}^4$ for the following reasons.

(a) First assume $d \ge g+4$. Since $\mathbb{P}^4$ has ∞^4 hyperplanes, the set $M'_{d,g}$ has dimension $\le 4d+4 < 5d+1-g$. Since each $C \subset W$ is contained in $U_{d,g}$, a dimensional count gives that each $C \in M_{d,g}$ contained in a general $W \in \mathcal{W}$ is non-degenerate. By [**3**] there is a non-empty open subset $V_{d,g}$ of $U_{d,g}$ such that all $C \in V_{d,g}$ have maximal rank in $\mathbb{P}^4$. A dimensional count gives that a general $W \in \mathcal{U}$ contains no element of $U_{d,g} \setminus V_{d,g}$.

(b) Now assume $d = g+3$. If $g \ne 3$, then every $C \in M_{d,g}$ is projectively normal, because $d \ge 2g+1$ and C is linearly embedded in a hyperplane of $\mathbb{P}^4$ ([**8**]). Now take any $C \in M_{6,3}$. Let $H \subset \mathbb{P}^4$ be the hyperplane spanned by C. Since $h^1(\mathcal{O}_C(1)) = 0$, the Castelnuovo-Mumford's lemma gives that C is projectively normal in H if and only if $h^1(H, \mathcal{I}_{C,H}(2)) = 0$, i.e. (Riemann-Roch) if and only if $h^0(H, \mathcal{I}_{C,H}(2)) = 0$. Assume $h^0(H, \mathcal{I}_{C,H}(2)) > 0$. Since $d > 4$, C is contained in a smooth quadric surface, S. Any smooth curve of a quadric surface cone is projectively normal ([**11**, Ex. V.2.9]). Thus S is smooth. Up to a choice of the ruling of S we may assume that $C \in |\mathcal{O}_S(2,4)|$. The curve $W \cap S \in |\mathcal{O}_S(5,5)|$ contains C. Let $E \in |\mathcal{O}_S(3,1)|$ be the curve linked to C by $W \cap S$. We have $\deg(E) = 4$. E has no multiple component, except at most lines in the ruling $|\mathcal{O}_S(1,0)|$. Each irreducible component of E_{red} is rational. If E is not irreducible, then two of its components meet. If E is irreducible, then it is a degree 4 rational curve not spanning $\mathbb{P}^4$. No $W \in \mathcal{U} \cap \mathcal{W}_1$ contains E.

2 - Non-degenerate curves

In this section we consider $C \in M_{d,g} \setminus M'_{d,g}$. By Remark 1 we may assume $d \in \{10, 11\}$. We saw in Section 1 that to prove Theorem 1 for all elements

$C \in M_{d,g} \setminus M'_{d,g}$ it is sufficient to exclude the ones with $h^1(\mathcal{I}_C(5)) > 0$.

Lemma 2. *There is no non-degenerate $C \in M_{d,g}$ with $9 \le d \le 11$, $1 \le g \le 3$, and $h^0(\mathcal{I}_C(2)) \ge 3$.*

Proof. Take a non-degenerate $C \in M_{d,g}$ with $9 \le d \le 11$, $1 \le g \le 3$, and $h^0(\mathcal{I}_C(2)) \ge 3$ and let G be the intersection of 2 general elements of $|\mathcal{I}_C(2)|$. G is a degree 4 complete intersection surface and $h^0(\mathcal{I}_G(2)) = 2$. Since $d > 8$ and $h^0(\mathcal{I}_C(2)) \ge 3$, there is an irreducible component $F \subsetneq G$ of G containing C. Since C is non-degenerate, F is non-degenerate and hence $\deg(F) \ge 3$. Thus $\deg(F) = 3$, i.e. F is a minimal degree non-degenerate surface of $\mathbb{P}^4$. By the classification of minimal degree surfaces in $\mathbb{P}^4$, either F is a cone over a rational normal curve $D \subset \mathbb{P}^3$ or F is isomorphic to the Hirzebruch surface F_1 embedded by the complete linear system $|h + 2f|$, where h is a section of the ruling of F_1 and f is a fiber of the ruling of F_1.

First assume that F is a cone and call o its vertex. Let $u : S \to F$ be the minimal resolution of F and let $C' \subset S$ be the strict transform of C. S is isomorphic to the Hirzebruch surface F_3 and u is induced by the complete linear system $|h+3f|$, where f is a fiber of the ruling of F_3 and h is the section of the ruling with negative self-intersection. We have $h^2 = -3$, $h \cdot f = 1$ and $f^2 = 0$ (intersection numbers). Since C' is smooth, u induces an isomorphism $C' \to C$ and hence C' has genus g. Take a, b with $C' \in |ah+bf|$. Since C' is not a line, we have $b \ge 3a > 0$. We have $d = (ah + bf) \cdot (h + 3f) = b$. Since $\omega_{F_3} \cong \mathcal{O}_{F_3}(-2h - 5f)$, the adjunction formula gives $\omega_{C'} \cong \mathcal{O}_{C'}((a-2)f + (d-5)f)$ and hence $2g - 2 = (ah+df) \cdot ((a-2)f + (d-5)f) = (d-3a)(a-2) + a(d-5)$. Since $g > 0$, we have $a \ge 2$. Since $d = b \ge 3a$, we get $2g - 2 \ge 2d - 10$, a contradiction.

Now assume $F \cong F_1$. Take $a, b \in \mathbb{N}$ such that $C \in |ah + bf|$. Since C is irreducible and not a line, we have $b \ge a > 0$. Since $\mathcal{O}_C(1) \cong \mathcal{O}_C(h + 2f)$, $h^2 = -1$, $h \cdot f = 1$, $f^2 = 0$ and $\deg(C) = d$, we have $d = a + b$. Since $\omega_{F_1} \cong \mathcal{O}_{F_1}(-2h-3f)$, the adjunction formula gives $\omega_C \cong \mathcal{O}_C((a-2)h+(b-3)f)$. Since $\deg(\omega_C) = 2g - 2$, we get $(ah + bf) \cdot ((a-2)h + (b-3)f) = 2g - 2$, i.e. $-a(a-2) + a(b-3) + b(a-2) = 2g - 2$, i.e $(b-a)(a-2) + a(b-3) = 2g-2$, i.e. $(d-2a)(a-2) + a(d-a-3) = 2g-2$. Since $g > 0$, we have $a \ge 2$. Since $d = b + a \ge 2a$, we get $2g - 2 \ge 2d - 10 \ge 8$, a contradiction. □

Remark 2. Fix $C \in M_{d,g} \setminus M'_{d,g}$, $d \le 11$. Let $H \subset \mathbb{P}^4$ be a general hyperplane. Since $H \cap C$ is in uniform position, we have $h^1(H, \mathcal{I}_{C\cap H,H}(3)) \le \max\{0, d-10\}$ and $h^1(H, \mathcal{I}_{C\cap H,H}(3)) = 1$ if and only if $d = 11$ and $C \cap H$ is contained in a rational normal curve of H ([**10**, Lemma 3.9]).

Remark 3. Fix a non-degenerate $C \in M_{d,g}$, $d \le 11$, $g > 0$, and assume the existence of a plane conic D with $\deg(D \cap C) \ge 10$. Let $\langle D \rangle$ be the plane spanned by D. Fix $q \in C \setminus C \cap \langle D \rangle$ and let H_q be the hyperplane spanned by $\langle D \rangle \cup \{q\}$. Since $d \le 11$ and C is non-degenerate, we get $d = 11$, $\deg(D \cap C) = 10$ and that $\{q\} = H_q \cap (C \setminus C \cap \langle D \rangle)$. The pencils of hyperplanes through $\langle D \rangle$ shows that C is rational, a contradiction.

Fix an integer $e \ge 6$. For any line $L \subset \mathbb{P}^4$ let $A(L, d, g, e)$ denote the set of all non-degenerate $X \in M_{d,g}$ such that $\deg(X \cap L) = e$. Let $A(d, g, e)$ be the union of all $A(L, d, g, e)$. Set $A'(d, g, e) := \cup_{f \ge e} A(d, g, e)$.

Lemma 3. *Either* $A(L, d, g, e) = \emptyset$ *or* $\dim A(L, d, g, e) \le 5d + 1 - g - 2e$ *or* $g = 3$, *C is hyperelliptic,* $d = e + 4$ *and* $\dim A(L, d, g, e) \le 5d + 1 - g - 2e + 2$.

Proof. Assume the existence of a non-degenerate $C \in M_{d,g}$ such that $\deg(L \cap C) = e$. Let $\ell : \mathbb{P}^4 \setminus L \to \mathbb{P}^2$ denote the linear projection from L. Let $C' \subset \mathbb{P}^4$ be the closure of $\ell(C \setminus C \cap L)$. Since C is non-degenerate, C' is non-degenerate, i.e. $\deg(C') \ge 2$. Since C is smooth, ℓ induces a morphism $u : C \to C'$ with $d - e = \deg(u) \cdot \deg(C')$. Since $g > 0$, either $\deg(u) > 1$ or $\deg(C') > 2$. We get $d - e \ge 3$ and that $d - e = 3$ only if $g = 1$. Hence we may assume $e \le d - 3$ and $e \le d - 4$ if $g = 2, 3$. Set $Z := C \cap L$. The vector space $H^0(N_C(-Z))$ is the tangent space at $[C]$ of the functor $A''(d, g, Z)$ of all $D \in M_{d,g}$ containing Z. Since L has ∞^e subschemes of degree e, we have $\dim A(L, d, g, e) \le e + \dim A''(d, g, Z')$ for some $Z' \subset L$ with $\deg(Z') = e$. We have $\chi(N_C(-Z)) = 5d + 1 - g - 3e$. Since $T\mathbb{P}^4$ is a quotient of $\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5}$ by the Euler's sequence, N_C is a quotient of $\mathcal{O}_C(1)^{\oplus 5}$. Hence if $d - e > 2g - 2$, then $h^1(N_C(-Z)) = 0$ and so $h^0(N_C(-Z)) = 5d + 1 - g - 3e$, concluding the proof unless $g = 3$ and $e = d - 4$. In this case we have $\deg(C') = 2$ and $\deg(u) = 2$ and so C is hyperelliptic. Let $M \subset \mathbb{P}^4$ be a general hyperplane containing L. We have $C \cap M = Z \cup S$, where S is a reduced set of 4 points (S is an element of $|\omega_C|$, because it is the inverse image by u of a general hyperplane section of C'). Since N_C is a quotient of $\mathcal{O}_C(1)^{\oplus 5}$, $N_C(-Z)$ is a quotient of $\mathcal{O}_C(S)^{\oplus 5}$. Hence $N_C(-Z)$ fits in an exact sequence

$$0 \to \mathcal{O}_C(S)^{\oplus 2} \to N_C(-Z) \to \mathcal{L} \to 0 \tag{1}$$

with $\mathcal{L}$ a line bundle of degree $5d + 2g - 2 - 3e - 8$. Since $\deg(\mathcal{L}) > 2g - 2$, we have $h^1(\mathcal{L}) = 0$. Since $h^1(\mathcal{O}_C(S)) = 1$, (1) gives $h^1(N_C(-Z)) \le 2$ and hence $h^0(N_C(-Z)) \le 5d + 1 - g - 3e + 2$. □

Lemma 4. *Assume* $C \in A'(d, g, 7)$, $d \le 11$, *and* $h^1(\mathcal{I}_C(5)) > 0$. *Then* $h^1(\mathcal{I}_C(3)) > h^1(\mathcal{I}_C(4)) > h^1(\mathcal{I}_C(5))$.

Proof. We prove only the first inequality, since the second one is similar. Let L be a line such that $\deg(L\cap C) \geq 7$. Since $d \leq 11$, we have $\deg(R\cap C) \leq 4$ for every line $R \neq L$. By Remark 3 there is no conic D with $\deg(D \cap C) \geq 10$. Let $N \subset \mathbb{P}^4$ be a plane with $N \cap L = \emptyset$. Set $V := H^0(\mathcal{I}_N(1))$. Let $M \subset \mathbb{P}^4$ be any hyperplane containing N. Since $L \nsubseteq M$, there is no line $R \subset M$ with $\deg(R \cap C) \geq 6$ and no conic $D \subset M$ with $\deg(D \cap C) \geq 10$. Hence $h^1(M, \mathcal{I}_{C\cap M,M}(4)) = 0$. Use Lemma 1. □

Lemma 5. *We have* $\dim A'(d, g, 7) \leq 5d+1-g-8+\tau$ *(resp.* $\dim A(d, g, 6) \leq 5d + 1 - 6 + \tau$*) with* $\tau = 2$ *if* $g = 3$ *and* $d = 11$ *(resp.* $d = 10$*) and* $\tau = 0$ *otherwise.*

Proof. Use Lemma 3 and that $\mathbb{P}^4$ has ∞^6 lines. □

Let $\Delta(11, g)$ be the set of all non-degenerate $C \in M_{11,g}$ such that for a general hyperplane $H \subset \mathbb{P}^4$ the set $C \cap H$ is contained in a rational normal curve of H.

Lemma 6. *Every irreducible component of* $\Delta(11, g)$ *has dimension* $\leq 46+g$.

Proof. Fix a hyperplane H, a rational normal curve $D \subset H$ and $S \subset D$ such that $\sharp(S) = 11$. The tangent space at $[C]$ of the functor $U(11, g, S)$ of all non-degenerate $C \in M_{11,g}$ containing S is isomorphic to $H^0(N_C(-S))$. We have $\chi(N_C(-S)) = 67 - g - 33$. Since $T\mathbb{P}^4$ is a quotient of $\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5}$, $N_C(-S)$ is a quotient of $\mathcal{O}_C(1)(-S)^{\oplus 5}$ and hence there is $\mathcal{A} \subset N_C(-S)$ with $\mathcal{A} \cong \mathcal{O}_C(1)(-S) \oplus \mathcal{O}_C(1)(-S)$ and $\mathcal{B} := N_C(-S)/\mathcal{A}$ torsion free. Since C is a smooth curve, $\mathcal{B}$ is a line bundle. Since $\deg(\mathcal{A}) = 0$, we have $\deg(\mathcal{B}) = 5d + 1 - g - 33 > 2g - 2$ and hence $h^1(\mathcal{B}) = 0$. Since $h^1(\mathcal{A}) = 2g$, we have $h^1(N_C(-S)) \leq 2g$ and hence $h^0(N_C(-S)) \leq 23 + g$. Since the set of all $S \subset D$ with $\sharp(S) = 11$ has dimension 11 and H contains ∞^{12} rational normal curves, we get the lemma. □

Proof [Proof of Theorem 1 for a non-degenerate $C \in M_{d,g}$]. Fix a non-degenerate $C \in M_{d,g}$. We assume $h^1(\mathcal{I}_C(5)) > 0$. If there is no line $L \subset \mathbb{P}^4$ with $\deg(L \cap C) \geq 6$, we have $h^1(\mathcal{I}_C(3)) \geq 8 + h^1(\mathcal{I}_C(5)) \geq 9$ and $h^1(\mathcal{I}_C(2)) \geq h^1(\mathcal{I}_C(3)) - \max\{0, d - 10\}$ (Lemma 1 and Remark 3). Hence $h^0(\mathcal{I}_C(2)) = h^1(\mathcal{I}_C(2)) + 14 + g - 2d \geq 23 + g - \max\{0, d - 10\} - 2d$. If either $d = 10$ or $d = 11$ and $g = 3$, we conclude by Lemma 2. Now assume $d = 11$, $g = 1, 2$. In all cases we have $h^0(\mathcal{I}_C(2)) > 0$, because $g > 0$. We have $5d+1-g-14-3d-g = 2d - 13 - g$.

Claim 1: Let Γ be the set of all $C \in M_{d,g} \setminus M'_{d,g}$ such that $h^0(\mathcal{I}_C(2)) \neq 0$. Then $\dim \Gamma \leq 3d + 14 + 2g$.

Proof of Claim 1: This proof is a modification of the case $g = 0$ ([**4**, Lemma 14]). The main new trick is the one used in the proof of Lemma 6. Since $\dim |\mathcal{O}_{\mathbb{P}^4}(2)| = 14$ and singular quadrics occur in codimension 1, it is sufficient to prove that for every smooth (resp., integral but singular) quadric Q the set Γ' of all $C \in M_{d,g}$ contained in Q has dimension $\le 3d+2g$ (resp., $\le 3d+1+2g$); in the first case we even prove that it has dimension $\le 3d + g$.

First assume that either Q is smooth or C does not intersect the singular locus V of Q. In this case the normal sheaf $N_{C,Q}$ is a rank 2 spanned vector bundle on C. Hence there is an inclusion $j : \mathcal{O}_C \to N_{C,Q}$ with $\mathcal{A} := N_{C,Q}/j(\mathcal{O}_C)$ a line bundle. Since $\deg(\mathcal{A}) = 3d + 2 - 2g > 2g - 2$, we have $h^1(\mathcal{A}) = 0$. Hence $h^1(N_{C,Q}) \le h^1(\mathcal{O}_C) = g$. Since $\det(N_{C,Q})$ has degree $3d - 2 + 2g$ and N_C has rank 2, Riemann-Roch gives $h^0(N_{C,Q}) \le 3d+g$, proving the Claim in this case.

Now assume $C \cap V \ne \emptyset$ and set $x := \deg(C \cap V)$. The vector space $H^0(\tau_Q)$ is the tangent space at the identity map of the automorphism group $\mathrm{Aut}(Q)$. Since $Q \setminus V$ is homogeneous, $\tau_Q|(Q \setminus V)$ is a spanned vector bundle. Since C is not a line and $\dim V \le 1$, the set $V \cap C$ is finite. Dualizing the natural map from the conormal sheaf of C in Q to Ω^1_Q we get a map $u : \tau_Q|C \to N_{C,Q}$ which is surjective outside the finite set $C \setminus C \cap V$. Since C is smooth and τ_Q is spanned at each point of $Q \setminus V$, there is an injective map $\ell : \mathcal{O}_C^2 \to N_{C,Q}$ with cokernel supported by finitely many points of C. Thus $h^1(N_{C,Q}) \le 2g$. Since we need to prove that $\dim \Gamma' \le 3d+1+2g$, it is sufficient to check this inequality when C is a general element of Γ'. In particular we may assume that $\deg(C' \cap V) = x$ for a general $C' \in \Gamma'$ and use induction on the integer x, the case $x = 0$ being true by the case $C \cap V = \emptyset$ proved before. Set $\Gamma'' := \{C' \in \Gamma' : \deg(V \cap C) = x\}$. It is sufficient to prove that $\dim \Gamma'' \le 3d + 1 + 2g$. Let $v : \widetilde{Q} \to Q$ be the blowing up of V, $E := v^{-1}(V)$ the exceptional divisor, and $\widetilde{C} \subset \widetilde{Q}$ the strict transform of C. Since C is smooth, v maps isomorphically $\widetilde{C}$ onto C and the numerical class of $\widetilde{C}$ with respect to $\mathrm{Pic}(\widetilde{Q})$ only depends on $\dim(V)$, d and x. Let Ψ be closure in $\mathrm{Hilb}(\widetilde{Q})$ of the strict transforms of all $C' \in \Gamma''$. It is sufficient to prove that $\dim \Psi \le 3d + 1 + 2g$. Take a general $D \in \Psi$. Since $\mathrm{Aut}(\widetilde{Q})$ acts transitively of $\widetilde{Q} \setminus E$, the first part of the proof gives $h^1(N_{D,\widetilde{Q}}) \le 2g$. Hence it is sufficient to prove that $\deg(N_{D,\widetilde{Q}}) \le 3d - 1$, i.e. $\deg(\tau_{\widetilde{Q}|D}) \le 3d + 1$, i.e. $\deg(\omega_{\widetilde{Q}}|D) \ge -3d - 1$. The group $\mathrm{Pic}(\widetilde{Q})$ is freely generated by E and the pull-back H of $\mathcal{O}_Q(1)$. We have $D \cdot H = d$ and $D \cdot E = x$. We have $\omega_{\widetilde{Q}} \cong \mathcal{O}_{\widetilde{Q}}(-3H + cE)$ with $c = -1$ if $\dim(V) = 0$ (see for instance [**12**], Example 8.5 (2)) and $c = 0$ if $\dim(V) = 1$ (see for instance [**12**], Example 8.5 (3)). Hence $\deg(\omega_{\widetilde{Q}|D}) = -3d + cx \ge -3d - 1$, concluding the proof of Claim 1.

(a) Now assume $C \in A'(d, g, 7)$.

(a1) First assume $(d, g) \ne (11, 3)$. We may assume $h^1(\mathcal{I}_C(5)) \ge 8$ (Lemma

5) and by Lemma 4 we get $h^1(\mathcal{I}_C(3)) \geq 10$. Hence $h^1(\mathcal{I}_C(2)) \geq 9$ (Remark 2). Thus $h^0(\mathcal{I}_C(2)) \geq 23 - 2d + g$. We conclude by Lemma 2, unless $d = 11$ and $g = 1$. In the latter case we conclude if $h^0(\mathcal{I}_C(2)) \geq h^1(\mathcal{I}_C(3))$. If $h^1(\mathcal{I}_C(2)) < h^1(\mathcal{I}_C(3))$, then $C \in \Delta(11, 1)$. By Lemma 6 we may assume $h^1(\mathcal{I}_C(5)) \geq 8$ and hence $h^1(\mathcal{I}_C(3)) \geq 10$ and $h^1(\mathcal{I}_C(2)) \geq 9$, i.e. $h^0(\mathcal{I}_C(2)) \geq 2$. By the proof of Lemma 2 C is contained in an irreducible surface T, which is the complete intersection of 2 quadric hypersurfaces. Fix a general hyperplane $H \subset \mathbb{P}^4$. Since $d > 8$, $C \cap H \subset T \cap H$, $T \cap H$ is irreducible and a rational normal curve of $\mathbb{P}^3$ is cut out by quadrics, Bezout implies $C \notin \Delta(11, 1)$, a contradiction.

(a2) Now assume $(d, g) = (11, 3)$. Lemma 5 gives $h^1(\mathcal{I}_C(5)) \geq 6$ and so we get $h^1(\mathcal{I}_C(3)) \geq 8$ and hence $h^1(\mathcal{I}_C(2)) \geq 7$. Thus $h^0(\mathcal{I}_C(2)) \geq 2$. By Lemma 2 we have $h^0(\mathcal{I}_C(2)) = 2$. Let $T \subset \mathbb{P}^4$ be the intersection of two different elements of $|\mathcal{I}_C(2)|$. Since $h^0(\mathcal{I}_C(2)) = 2$, T is a degree 4 irreducible surface. Since $h^1(\mathcal{I}_C(3)) \geq 8$, we have $h^0(\mathcal{I}_C(3)) \geq 12$. Hence the natural map $H^0(\mathcal{I}_C(2)) \otimes H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \rightarrow H^0(\mathcal{I}_C(3))$ is not surjective. Take $Y \in |\mathcal{I}_C(3)|$ not containing T. Since $T \cap Y$ is a degree 12 complete intersection curve containing C and $d = 11$, $T \cap Y$ links C to a line and so C is arithmetically normal ([**19**, Theorem 5.3.1]). In particular $h^1(\mathcal{I}_C(5)) = 0$, a contradiction.

(b) Now assume $C \in A(d, g, 6)$ and $C \notin A'(d, g, 7)$. By Lemma 5 we may assume $h^1(\mathcal{I}_C(5)) \geq 4 + \beta$ with $\beta = 0$ if $(d, g) = (10, 3)$ and $\beta = 2$ otherwise. Since $C \notin A'(d, g, 7)$, we get $h^1(\mathcal{I}_C(4)) \geq 8 + \beta$.

Claim 2: There is a unique line $L \subset \mathbb{P}^4$ such that $C \in A(L, d, g, 6)$.

Proof of Claim 2: Assume the existence of lines L, R such that $C \in A(L, d, g, 6) \cap A(R, d, g, 6)$ and $L \cap R$. Since $d \leq 11$, we have $L \cap R \neq \emptyset$. Let $N \subset \mathbb{P}^4$ be the plane spanned by $L \cup R$. Since $\deg(L \cap R) = 1$, we get $\deg(N \cap C) \geq 11$ and so C is degenerate, a contradiction.

Let $E \subset \mathbb{P}^4$ be any plane such that $E \cap L = \emptyset$. No hyperplane $M \subset \mathbb{P}^4$ containing E contains L. By Remark 3 we have $h^1(M, \mathcal{I}_{C \cap M}(4)) = 0$ for every $M \in |\mathcal{I}_E(1)|$. Set $V := H^0(\mathcal{I}_E(1))$. Lemma 1 gives $h^1(\mathcal{I}_C(3)) > h^1(\mathcal{I}_C(4))$ and hence $h^1(\mathcal{I}_C(2)) \geq 8 + \beta$. Thus $h^0(\mathcal{I}_C(2)) \geq 3$, contradicting Lemma 2. □

3 - Degenerate curves

In this section we consider curves in $M'_{d,g}$. By Remark 1 we may assume $d \geq 9$. We only need to consider $C \in M'_{d,g}$ contained in some $W \in \mathcal{W}_1$.

Each such curve C spans a hyperplane H and $h^1(\mathcal{O}_C(1)) = 0$. Since $h^1(\mathcal{O}_C(1)) = 0$ for all $C \in M_{d,g}(H)$, $M_{d,g}(H)$ is smooth, irreducible and of dimension $4d$. Since $\mathbb{P}^4$ has ∞^4 hyperplanes, to check that a general $W \in \mathcal{W}$

contains no $C \in M'_{d,g}$ it is sufficient to check all $C \in M'_{d,g}$ with $h^1(\mathcal{I}_C(5)) \geq 5d+1-g-4d+4 = d-3-g$. However, since a general $C \in M_{d,g}(H)$ has $h^1(H, \mathcal{I}_C(5)) = 0$ ([**2**] and the inequality $5d+1-g \leq \binom{8}{3}$) we only need to check and exclude a set of codimension at least $d-2-g$ of $M_{d,g}$ and so we may assume that $h^1(\mathcal{I}_C(5)) \geq d-2-g$. Fix $C \in M'_{d,g}$ with $h^1(\mathcal{I}_C(5)) \geq d-2-g$ and let $H \subset \mathbb{P}^4$ be the hyperplane spanned by C. Note that $h^1(H, \mathcal{I}_{C,H}(t)) = h^1(\mathcal{I}_C(t))$ for all t. Let α be the minimal degree of a surface of H containing C. Since $h^0(H, \mathcal{O}_H(5)) = 56$, we have $h^0(H, \mathcal{I}_{C,5}(5)) \geq 56+d-2-g-5d-1+g = 53-4d \geq 9$. Hence $\alpha \leq 5$. Since C is irreducible and $h^0(\mathcal{I}_{C,H}(\alpha-1)) = 0$, every degree α surface containing C is irreducible. Let $S \subset H$ be a degree α surface containing C.

By [**10**, Lemma 3.9] we get the following lemma.

Lemma 7. *Let $N \subset H$ be a general plane. We have $h^1(N, \mathcal{I}_{C\cap N,N}(t)) \leq \max\{0, d-2t-1\}$ and $h^1(N, \mathcal{I}_{C,N}(t)) = d-2t-1 > 0$ if and only if $N \cap C$ is contained in a conic.*

Remark 4. Let $N \subset H$ be a plane. Fix and integer $t \geq 2$. Since $\dim(C \cap N) = 0$, we have $h^2(N, \mathcal{I}_{C\cap H,N}(t)) = h^2(N, \mathcal{O}_N(t)) = 0$ and hence the exact sequence

$$0 \to \mathcal{I}_{C,H}(t-1) \to \mathcal{I}_{C,H}(t) \to \mathcal{I}_{C\cap N,N}(t) \to 0. \tag{2}$$

gives $h^1(H, \mathcal{I}_{C,H}(t-1)) \geq h^1(H, \mathcal{I}_{C,H}(t)) - h^1(N, \mathcal{I}_{C\cap N,N}(t))$. Now assume that N is general. By Lemma 7 we have $h^1(N, \mathcal{I}_{C\cap N,N}(t)) = 0$ if either $d \leq 2t+1$ or $d = 2t+2$ and $N \cap C$ is not contained in a conic. Since $d \leq 11$, we alway have $h^1(H, \mathcal{I}_{C,H}(4)) \geq h^1(H, \mathcal{I}_{C,H}(5))$.

Lemma 8. *We have $\alpha \geq 3$. If $\alpha = 3$, then $h^0(H, \mathcal{I}_{C,H}(3)) = 1$*

Proof. First assume $\alpha = 2$. Since $d > 4$, C is contained in a unique quadric, S. Any smooth curve E of a quadric surface cone is projectively normal ([**11**, Ex. V.2.9]) and in particular $h^1(\mathcal{I}_E(5)) = 0$. Thus S is a smooth quadric. Take $a, b \in \mathbb{N}$ such that $C \in |\mathcal{O}_S(a,b)|$ with, say, $a \leq b$. We have $d = a+b$ and $g = ab-a-b+1 = ad-a^2-d+1 = a(d-a)-d+1$. Since $g > 0$, we have $a \geq 2$ and hence $g \geq 5$, a contradiction.

Now assume $\alpha = 3$ and $h^0(H, \mathcal{I}_{C,H}(3)) \geq 2$. Take a degree 3 surface $S' \subset H$ with $S \neq S'$ and $C \subset S'$. Since S, S' are integral, $C \subseteq S \cap S'$ and $d \geq 9$, we have $d = 9$ and $C = S \cap S'$. C is not a complete intersection of 2 cubic surfaces of H, because it is not arithmetically normal, since $h^1(\mathcal{I}_C(5)) > 0$. $\square$

R e m a r k 5. Fix an integer $e \ge 6$, a line $L \subset H$, a zero-dimensional scheme $Z \subset L$ with $\deg(Z) = e$. Let $A(H, Z, d, g)$ be the scheme of all $X \in M_{d,g}(H)$ containing Z. Fix $X \in A(H, Z, d, g)$. The vector space $H^0(N_{C,X}(-Z))$ is the Zariski tangent space of $A(H, Z, d, g)$ at $[X]$ and $\deg(N_{X,H}(-Z)) = 4d+2g-2-2e$. Thus $\chi(N_{C,X}(-Z)) = 4d - 2e$. Take a general plane $N \subset H$ containing L. By Bertini's theorem the scheme $N \cap X$ is the union of Z and a set $E \subset X \backslash X \cap L$. Hence $\mathcal{O}_X(-Z) \cong \mathcal{O}_X(1)(-E)$. We have $\sharp(E) = d - e$. Set $\eta := h^1(\mathcal{O}_X(E))$. Since $TH(-1)$ is a quotient of $\mathcal{O}_H^{\oplus 4}$ by the Euler's sequence, $N_{X,H}(-Z)$ is quotient of $\mathcal{O}_X(E)^{\oplus 4}$. Hence there is a inclusion $j : \mathcal{O}_X(E) \to N_{X,H}(-Z)$ with $N_{X,H}(-Z)/j(\mathcal{O}_X(E))$ a line bundle. Since $\deg(N_{X,H}(-Z)/j(\mathcal{O}_X(E))) = 4d + 2g - 2 - 3e > 2g - 2$, we have $h^1(N_{X,H}(-Z)/j(\mathcal{O}_X(E))) = 0$. Hence $h^0(N_{X,H}(-Z)/j(\mathcal{O}_X(E))) \le 4d - 2e + \eta$. Since $g > 0$, the pencil of all planes of of H containing L gives $d - e \ge 2$ and $h^0(\mathcal{O}_C(E)) \ge 2$. Hence $\eta = 0$ if either $g = 1$ or $g = 2$ and $d - e \ge 3$ or $g = 3$ and $d - e \ge 5$, $\eta \le 1$ if either $g = 2$ and $d - e = 2$ or $g = 3$ and $d - e \ge 3$, $\eta \le 2$ if $g = 3$ and $d - e = 2$.

R e m a r k 6. Let $\Delta'(d, g, e)$, $e \ge 6$, be the set of all $C \in M'_{d,g}$ contained in an element $W \in \mathcal{W}_1$ and such that there is a a line $L \subset \mathbb{P}^4$ with $\deg(L \cap C) = e$. For any line $L \subset \mathbb{P}^4$ let $A(L, d, g, e)_1$ denote the set of all $C \in M'_{d,g}$ such that $\deg(L \cap C) = e$. We have $\dim A(L, d, g, e) \le 4d - e + \eta + 2$ (with η as in Remark 5) because L has ∞^e degree e subschemes and $\mathbb{P}^4$ have ∞^2 hyperplanes containing L. Since $e \ge 6$, L is contained in any $W \in \mathcal{W}_1$ such that $W \supset C$ by Bezout W. Since on each $W \in \mathcal{W}_1$ there are finitely many lines, to prove that a general $W \in \mathcal{W}$ contains no element of $\Delta'(d, g, e)$ it is sufficient to test the curves $C \in M'_{d,g}$ with $h^1(\mathcal{I}_C(5)) \ge d - 1 - g - \eta + e$.

Fix a conic $D \subset H$. Let $A(d, g, H, D, 10)$ be the set all $X \in M_{d,g}(H)$ contained in some $W \in \mathcal{W}_1$ such that there is a conic D with $\deg(D \cap X) \ge 10$. If $D \subset W$, then D is smooth by the definition of $\mathcal{W}_1$. If $D \nsubseteq W$, then $\deg(D \cap W) = \deg(X \cap D) = 10$ and $X \cap D \in |\mathcal{O}_D(5)|$.

L e m m a 9. *If* $d = 10$ (*resp.* $d = 11$) *the set* $A(d, g, H, D, 10)$ *has dimension* $\le 4d - 10 + \gamma$, *where* $\gamma = g$ *if* $d = 10$ *and* $\gamma = g - 1$ *if* $d = 11$.

P r o o f. Fix $X \in A(d, g, H, 10)$ and let $D \subset H$ be a conic with $\deg(D \cap X) \ge 10$. Fix a curvilinear $Z \subseteq D \cap X$ with $\deg(Z) = 10$. Let $A(d, g, H, Z)$ be the the set of all $Y \in M_{d,g}(H)$ containing Z. The vector space $H^0(N_{X,H}(-Z))$ is the tangent space of $A(d, g, H, Z)$ at $[X]$. We have $\deg(N_{X,H}(-Z)) = 4d+2g-2-20$ and $\chi(N_{X,H}(-Z)) = 4d - 20$. First assume $d = 10$, i.e. $Z \in |\mathcal{O}_X(1)|$. Since $N_{X,H}(-1)$ is spanned, there is $\mathcal{A} \subset N_{X,H}(-1)$ with $\mathcal{A} \cong \mathcal{O}_X$ and $N_{X,H}(-Z)/\mathcal{A}$ locally free. Since $\deg(N_{X,H}(-Z)/\mathcal{A}) = 4d + 2g - 2 - 20 > 2g - 2$, we have $h^1(N_{X,H}(-Z)/\mathcal{A}) = 0$ and hence $h^1(N_{X,H}(-Z)) \le h^1(\mathcal{A}) = g$. Now assume

$d = 11$. Since X is a smooth curve, there is a unique $q \in X$ such that $q + Z = X \cap N$ as effective divisors of X, where N is the plane spanned by D. In this case we find $\mathcal{B} \subset N_{X,H}(-1)$ with $\mathcal{B} \cong \mathcal{O}_X(q)$ and $N_{X,H}(-Z)/\mathcal{B}$ locally free. Since $\deg(N_{X,H}(-Z)/\mathcal{B}) = 4d + 2g - 2 - 21 > 2g - 2$, we get $h^1(N_{X,H}(-Z)) \le h^1(\mathcal{B}) = g-1$. If D is smooth, then it has ∞^{10} schemes of degree 10. If D is not smooth, then we use that $X \subset W$ for some $W \in \mathcal{W}_1$ and that $\dim |\mathcal{O}_D(5)| = 10$. In both cases we get $\dim A(d, g, H, 10) \le 4d - 10 + \gamma$. □

Proof [Proof of Theorem 1 for a degenerate C]. We saw that to prove that a general $W \in \mathcal{W}$ contains no degenerate element of $M_{d,g}$ it is sufficient to exclude all $C \in M_{d,g}(H)$ with $h^1(H, \mathcal{I}_{d,g}(5)) \ge d - 2 - g$. Set $x := h^1(H, \mathcal{I}_{C,H}(3)) - h^1(\mathcal{I}_{C,H}(5))$. We have $h^0(H, \mathcal{I}_{C,H}(3)) = 20 - 3d - 1 + g - h^1(H, \mathcal{I}_{C,H}(3)) \ge 19 - 3d + g + x - d + 2 - g = 17 - 2d + x$. Hence by Lemma 8 we conclude if $x \ge 2d - 15$.

(a) Assume for the moment the non-existence of a line L with $\deg(L \cap C) \ge 6$ and the non-existence of a conic D with $\deg(D \cap C) \ge 10$. Since $d < 12$, by [**7**, Corollary 2 or Remarques (i)] we have $h^1(N, \mathcal{I}_{C \cap N,N}(t)) = 0$ for all $t \ge 4$. Lemma 1 gives $h^1(H, \mathcal{I}_{C,H}(3)) \ge 3 + h^1(H, \mathcal{I}_{C,H}(4)) \ge 6 + h^1(H, \mathcal{I}_{C,H}(5))$. Hence $x \ge 6$ and in particular we conclude if $d \le 10$. Now assume $d = 11$. Since $x \ge 6$, we have $h^0(\mathcal{I}_{C,H}(3)) > 0$ and hence $h^0(H, \mathcal{I}_{C,H}(5)) \ge 10$, i.e. $h^1(H, \mathcal{I}_{C,H}(5)) \ge 10 - g$. Hence $h^1(H, \mathcal{I}_{C,H}(3)) \ge 16 - g$ and so $h^0(H, \mathcal{I}_{C,H}(3)) \ge 2$, concluding by Lemma 8.

(b) Now assume the existence of a line $L \subset H$ such that $e := \deg(L \cap C) \ge 7$. By Remark 6 we may assume $h^1(H, \mathcal{I}_{C,H}(5)) \ge d + 6 - g - \eta$ with η associated to the integer $e = 7$ and hence $\eta = 1$ if $(d, g) = (11, 3)$ and $\eta = 0$ otherwise. Lemma 1 $h^1(H, \mathcal{I}_{C,H}(4)) \ge d + 7 - g - \eta$. Since $h^1(N, \mathcal{I}_{C \cap N,N}(4)) \le 2$, (2) gives $h^1(H, \mathcal{I}_{C,H}(3)) \ge d + 5 - g - \eta$ and so $h^0(\mathcal{I}_{C,H}(3)) \ge 24 - 2d - \eta$. We conclude by Lemma 8, unless $d = 11$ and $\eta = 1$, i.e. $(d, g) = (11, 3)$. If $(d, g) = (11, 3)$ we get $\alpha = 3$. Let $S \subset H$ be the only irreducible degree 3 surface containing C (Lemma 8). Since C is not a line, it is not contained in the singular locus of S. Fix a general plane $N \subset H$. Since $S \cap N$ is a irreducible plane cubic and $C \cap N$ is contained in the smooth locus of $S \cap N$, we have $h^1(N, \mathcal{I}_{C \cap N,N}(4)) = 0$. Hence (2) gives $h^1(H, \mathcal{I}_{C,H}(3)) \ge d + 7 - g - \eta$ and so $h^0(H, \mathcal{I}_{C,H}(3)) \ge 3$, a contradiction.

(c) Now assume the existence of a line $L \subset H$ such that $\deg(L \cap C) = 6$, but that there is no line R with $\deg(R \cap C) \ge 7$. By Remarks 6 we may assume $h^1(\mathcal{I}_C(5)) \ge d + 5 - g$. Since there is no line R with $\deg(R \cap C) \ge 7$, Lemma 1 gives $h^1(M, \mathcal{I}_{C,M}(4)) \ge d + 8 - g$. We conclude as in step (b).

(d) To conclude the proof in the degenerate case it is sufficient to handle the case in which there is a conic D with $\deg(D \cap C) \ge 10$ and in particular $d \in$

$\{10, 11\}$. By steps (b) we may assume that there is no line $L \subset M$ with $\deg(L \cap C) \geq 7$ and so we may assume that $h^1(M, \mathcal{I}_{C,M}(4)) \geq 3 + h^1(M, \mathcal{I}_{C,M}(5))$ (Lemma 1).

(d1) In this step we handle the case in which D is not contained in the quintic hypersurface $W \in \mathcal{W}_1$ which by assumption contains C. Let A be the plane spanned by D. Since W is smooth, its Picard group is generated by $\mathcal{O}_W(1)$ and in particular $A \nsubseteq W$. Hence $A \cap W$ is a plane quintic. Since $D \nsubseteq W$, we get $\deg(D \cap W) = 10 = \deg(D \cap (A \cap W))$. Hence $\deg(D \cap C) = 10$ and $A \cap C = D \cap C$. Thus $d = 10$ and $Z := D \cap C \in |\mathcal{O}_C(1)|$. We have $N_{C,M}(-Z) \cong N_{C,M}(-1)$. The Euler sequence of TM shows that $N_{C,M}$ is a quotient of $\mathcal{O}_C^{\oplus 4}(1)$. Thus $N_{C,M}(-1)$ is spanned. Therefore there is $\mathcal{A} \subset N_{C,M}(-1)$ with $\mathcal{A} \cong \mathcal{O}_C$ and $N_{C,M}(-1)/\mathcal{A}$ torsion free and so a line bundle. Since $\deg(N_{C,M}(-1)) = 2d + 2g - 2 = 18 + 2g$, we have $h^1(N_{C,M}(-1)/\mathcal{A}) = 0$ and so $h^1(N_{C,M}(-1)) \leq h^1(\mathcal{O}_C) = g$. Hence $h^0(N_C(-1)) = h^0(\mathcal{O}_C) + h^0(N_{C,M}(-1)) \leq 21 + g$. There are ∞^6 planes $A \subset \mathbb{P}^4$ and (for a fixed W) each of them is associated to a unique Z. So we may assume $h^1(\mathcal{I}_C(5)) \geq 24 - 2g$.

(d2) Now we assume that D is contained in the quintic hypersurface $W \in \mathcal{W}_1$ containing C. By the definition of $\mathcal{W}_1$, D is a smooth conic. Thus D has ∞^{10} zero-dimensional schemes of degree 10. Since W only contains finitely many conics, we conclude as in Remark 6 using smooth conics instead of lines and quoting Remark 6. □

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