Riv. Mat. Univ. Parma, Vol. **8** (2017), 287-306

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Existence and non-existence of functional solutions for the equations of axially symmetric gravitational fields in general relativity

Abstract. A theorem of existence and non-existence of solutions for a boundary value problem for the equations of axially symmetric gravitational field in vacuum is given using the method of functional solutions. The boundary value problem is reduced to a two-point problem for a Bernoulli equation. Conditions are given under which solutions exist or not exist.

Keywords. Axially symmetric gravitational fields, existence and uniqueness of solutions, Weyl-Lewis-Papapetrou coordinates.

Mathematics Subject Classification (2010): 83C10, 83C05.

1 - Introduction

The stationary Einstein's field equations do not fit clearly in any of the types in which the partial differential equations are usually classified. They are nonlinear with the added complication to be over determined. Particularly difficult is the problem of determining the boundary conditions which must be added to the system in order to have a well-posed boundary value problem [**3**]. In this paper we study a problem for the Einstein's equations in vacuum assuming an axially symmetric situation. Stationary axially symmetric solutions are of great astrophysical importance: they describe the exterior of bodies like stars, galaxies or accretion disks in equilibrium. We refer to [**2**] and to [**1**] for a comprehensive survey. We adopt here the formulation given by T. Lewis [**5**] and prove that the corresponding boundary value problem has one and only one

Received: September 14, 2016; accepted in revised form: June 20, 2017.

solution for certain boundary conditions and no solutions for others. In the first and second Sections, after recalling how to derive the Weyl-Lewis-Papapetrou coordinate system, a useful parametrization is used to obtain a reformulation of the problem which uses only two independent partial differential equations. This material is largely, but not entirely, contained in [**5**] and [**6**]. The method used in this paper is not the only possible, for various different approaches we refer to the book [**9**]. The reduced system of P.D.E is studied in the remaining sections using the method of the functional solutions. We solve a two-point problem for a Bernoulli's ordinary differential equation and obtain, as a corollary, the proposed result of existence and non-existence of solutions.

2 - Derivation of the governing equations

The Einstein's equations in vacuum, corresponding to the metric

$$ds^2 = f dx_0^2 - e^{\mu} dx_1^2 - e^{\mu} dx_2^2 - l dx_3^2 - 2m dx_0 dx_3, \tag{2.1}$$

were studied by T. Lewis [**5**], who assumed an axially symmetric situation. The last term in (2.1) corresponds to the case of a rotating mass. If $m = 0$ in (2.1) we have the Weyl metric [**10**]. The field depends, in cylindrical coordinates, upon two variables x_1 and x_2, $x_2 = 0$ being the axis of symmetry of the field. x_0 is the time coordinate and x_3 an angular variable varying from 0 to 2π. The gravitational field equations will be derived from a variational principle. The relevant lagrangian is

$$2G = \Gamma^r_{ik} \frac{\partial (g^{ik} \sqrt{|g|})}{\partial x_r} - \Gamma^r_{ir} \frac{\partial (g^{ik} \sqrt{|g|})}{\partial x_k}, \qquad (^1) \tag{2.2}$$

(see [**10**] and, for a more modern treatment, [**4**]), where

$$\begin{gathered} g^{00} = r^{-2} l, \ g^{11} = -e^{-\mu}, \ g^{22} = -e^{-\mu}, \\ g^{33} = r^{-2} f, \ g^{03} = r^{-2} m, \ |g| = r^2 e^{2\mu} \end{gathered} \tag{2.3}$$

and

$$r^2 = fl + m^2. \tag{2.4}$$

[1] Use is made of the summation convection.

All the others g_{ij} and g^{ij} are zero. Calculating the Chistoffel symbols which appear in (2.2) we find

$$G = \frac{[f,l]+[m,m]}{2r} + r_{,i}\mu_{,i} \qquad (^2) \tag{2.5}$$

where

$$[f,l] = f_{,i}l_{,i} \quad [m,m] = m_{,i}m_{,i}.$$

This lagrangian can also be written

$$G = \frac{[f,l]+[m,m]}{2r} + \mu_{,i}\frac{f_{,i}l + fl_{,i} + 2mm_{,i}}{2r} \tag{2.6}$$

since

$$r_{,i} = \frac{f_{,i}l + fl_{,i} + 2mm_{,i}}{2r}.$$

We find, as lagrangian binomial corresponding to $f = g_{00}$,

$$\Big(\frac{\partial G}{\partial f_{,i}}\Big)_{,i} - \frac{\partial G}{\partial f} = \frac{1}{2}\Big(\frac{l_{,i}}{r}\Big)_{,i} + \frac{l}{4r}\Big(\frac{[f,l]+[m,m]}{r^2} + 2\nabla^2\mu\Big), \tag{2.7}$$

where $\nabla^2\mu = \mu_{,ii}$ denotes the laplacian. In the same way we have, for the lagrangian binomial corresponding to $l = g_{33}$,

$$\Big(\frac{\partial G}{\partial l_{,i}}\Big)_{,i} - \frac{\partial G}{\partial l} = \frac{1}{2}\Big(\frac{f_{,i}}{r}\Big)_{,i} + \frac{f}{4r}\Big(\frac{[f,l]+[m,m]}{r^2} + 2\nabla^2\mu\Big)$$

and for $m = g_{03}$

$$\Big(\frac{\partial G}{\partial m_{,i}}\Big)_{,i} - \frac{\partial G}{\partial m} = \frac{1}{2}\Big(\frac{m_{,i}}{r}\Big)_{,i} + \frac{m}{4r}\Big(\frac{[f,l]+[m,m]}{r^2} + 2\nabla^2\mu\Big). \tag{2.8}$$

To compute the lagrangian binomial corresponding to g_{11} we proceed in a slightly different way. Let $\gamma = g_{11} = -e^{\mu}$, thus $\mu_{,i} = \dfrac{\gamma_{,i}}{\gamma}$. The lagrangian then becomes

$$G = \frac{1}{2}\Big(\frac{f_{,i}l_{,i} + m_{,i}^2}{r}\Big) + r_{,i}\frac{\gamma_{,i}}{\gamma}. \tag{2.9}$$

We have

$$\frac{\partial G}{\partial \gamma_{,i}} = \frac{r_{,i}}{\gamma}, \quad \Big(\frac{\partial G}{\partial \gamma_{,i}}\Big)_{,i} = e^{-\mu}(\mu_{,i}r_{,i} - r_{,ii}), \quad \frac{\partial G}{\partial \gamma} = e^{-\mu}r_{,i}\mu_{,i}.$$

[2] Here and hereafter we use for the partial derivative with respect to x_1 and x_2 the notation $\frac{\partial f}{\partial x_i} = f_{,i}, \quad i = 1, 2.$

Hence

$$\left(\frac{\partial G}{\partial \gamma_{,i}}\right)_{,i} - \frac{\partial G}{\partial \gamma} = -e^{-\mu} r_{,ii}. \tag{2.10}$$

Equating to zero the lagrangian binomials, we arrive, for the determination of f, l, m and μ, to the following highly symmetrical, but quite complex system of partial differential equations

$$\nabla^2 r = 0 \tag{2.11}$$

$$\left(\frac{f_{,i}}{r}\right)_{,i} + \frac{f}{2r}\left(\frac{[f,l]+[m,m]}{r^2} + 2\nabla^2\mu\right) = 0 \tag{2.12}$$

$$\left(\frac{l_{,i}}{r}\right)_{,i} + \frac{l}{2r}\left(\frac{[f,l]+[m,m]}{r^2} + 2\nabla^2\mu\right) = 0 \tag{2.13}$$

$$\left(\frac{m_{,i}}{r}\right)_{,i} + \frac{m}{2r}\left(\frac{[f,l]+[m,m]}{r^2} + 2\nabla^2\mu\right) = 0 \tag{2.14}$$

$$r^2 = fl + m^2. \tag{2.15}$$

The important fact that $r(x_1, x_2)$ is an harmonic function was first noticed by E. Weyl in [**10**]. Moreover, the equations (2.12), (2.13) and (2.14) are invariant under conformal mapping. On the other hand, if $z(x_1, x_2)$ is the harmonic conjugate of $r(x_1, x_2)$ the analytic function $z(x_1, x_2) + ir(x_1, x_2)$ defines a conformal mapping from the complex plane $x_1 + ix_2$ to $z + ir$. This is the point of view of the Weyl-Lewis-Papapetrou coordinate systems, see [**7**] and [**8**], in which one takes r and z as independent variables. The system (2.12), (2.13) and (2.14) retains the same form, but with a new meaning of r. We write it in a more convenient form as follows:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial f}{\partial z}\right) + \frac{f}{2r}\left(\frac{[f,l]+[m,m]}{r^2} + 2\nabla^2\mu\right) = 0 \tag{2.16}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial l}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial l}{\partial z}\right) + \frac{l}{2r}\left(\frac{[f,l]+[m,m]}{r^2} + 2\nabla^2\mu\right) = 0 \tag{2.17}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial m}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial m}{\partial z}\right) + \frac{m}{2r}\left(\frac{[f,l]+[m,m]}{r^2} + 2\nabla^2\mu\right) = 0, \tag{2.18}$$

where now $[f,l] = \frac{\partial f}{\partial r}\frac{\partial l}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial l}{\partial z}$ and $\nabla^2\mu = \frac{\partial^2\mu}{\partial r^2} + \frac{\partial^2\mu}{\partial z^2}$. Let us multiply (2.17) by f, (2.16) by l and (2.18) by m. Adding the resulting equations we obtain, after simple calculations,

$$\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial r^2}{\partial r}\Big)+\frac{\partial}{\partial z}\Big(\frac{1}{r}\frac{\partial r^2}{\partial z}\Big)=\frac{1}{r}([f,l]+[m,m])+2r\nabla^2\mu. \tag{2.19}$$

On the other hand,

$$\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial r^2}{\partial r}\Big)+\frac{\partial}{\partial z}\Big(\frac{1}{r}\frac{\partial r^2}{\partial z}\Big)=0. \tag{2.20}$$

Hence (2.19) becomes

$$\nabla^2\mu = \frac{1}{2r^2}\big([f,l]+[m,m]\big). \tag{2.21}$$

Therefore (2.17), (2.16) and (2.18) simplify to

$$\frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{r}\frac{\partial f}{\partial r} = -\frac{f}{r^2}([f,l]+[m,m]) \tag{2.22}$$

$$\frac{\partial^2 l}{\partial r^2} + \frac{\partial^2 l}{\partial z^2} - \frac{1}{r}\frac{\partial l}{\partial r} = -\frac{l}{r^2}([f,l]+[m,m]) \tag{2.23}$$

$$\frac{\partial^2 m}{\partial r^2} + \frac{\partial^2 m}{\partial z^2} - \frac{1}{r}\frac{\partial m}{\partial r} = -\frac{m}{r^2}([f,l]+[m,m]). \tag{2.24}$$

The system (2.22), (2.23) and (2.24) does not depend on μ. Therefore, if we know f, l and m we can obtain μ solving, with suitable boundary conditions, (2.21) as a Dirichlet's problem. Moreover, in view of the condition

$$fl + m^2 = r^2 \tag{2.25}$$

the three equations (2.22), (2.23), (2.24) are not independent. It would, therefore, be desirable to have a new formulation with only two equations in two unknowns in which (2.25) is automatically satisfied. To this end we can use the parametrization of f, l and m

$$\begin{cases} f = r(\cosh\lambda - \cosh(2u)\sinh\lambda) \\ l = r(\cosh\lambda + \cosh(2u)\sinh\lambda) \\ m = r\sinh\lambda\sinh(2u) \end{cases} \tag{2.26}$$

which can also be written

$$(2.27)\qquad \begin{cases} f &= r(e^{-\lambda}\cosh^2 u - e^{\lambda}\sinh^2 u)\\ l &= r(e^{\lambda}\cosh^2 u - e^{-\lambda}\sinh^2 u)\\ m &= r(e^{\lambda} - e^{-\lambda})\sinh u\cosh u. \end{cases}$$

The condition (2.25) is satisfied by (2.26) or (2.27). In terms of λ and u we obtain the relatively simple lagrangian

$$(2.28)\qquad G = -\frac{r}{2}\Big[\Big(\frac{\partial\lambda}{\partial r}\Big)^2+\Big(\frac{\partial\lambda}{\partial z}\Big)^2\Big]+2r\sinh^2\lambda\Big[\Big(\frac{\partial u}{\partial r}\Big)^2+\Big(\frac{\partial u}{\partial z}\Big)^2\Big]+\frac{1}{2r}+\frac{\partial\mu}{\partial r}.$$

For a closely related parametrization and for the corresponding lagrangian we refer to R. A. Matzner and C. W. Misner [**6**]. The two last terms of the lagrangian (2.28) do not contribute to the Euler equations, which are

$$(2.29)\qquad -\Big[\frac{\partial}{\partial r}\Big(r\frac{\partial\lambda}{\partial r}\Big)+\frac{\partial}{\partial z}\Big(r\frac{\partial\lambda}{\partial z}\Big)\Big]= 2r\sinh(2\lambda)\Big[\Big(\frac{\partial u}{\partial r}\Big)^2+\Big(\frac{\partial u}{\partial z}\Big)^2\Big]$$

$$(2.30)\qquad \frac{\partial}{\partial r}\Big(r\sinh^2\lambda\frac{\partial u}{\partial r}\Big)+\frac{\partial}{\partial z}\Big(r\sinh^2\lambda\frac{\partial u}{\partial z}\Big)= 0.$$

The system (2.29), (2.30) represents a considerable simplification with respect to the previous formulation (2.22)-(2.25). Unfortunately, the parametrization (2.26) has the disadvantage of not defining a global diffeomorphism. This question is studied in the next section.

3 - The range of (3.1)

The mapping $(f,l) = \mathcal{F}(\lambda,u;r)$ from $\mathbf{R}^2$ to $\mathbf{R}^2$, depending on the parameter r,

$$(3.1)\qquad \begin{cases} f = r(e^{-\lambda}\cosh^2 u - e^{\lambda}\sin^2 u)\\ l = r(e^{\lambda}\cosh^2 u - e^{-\lambda}\sinh^2 u) \end{cases}$$

or equivalently

$$(3.2)\qquad \begin{cases} f = r(\cosh\lambda - \cosh(2u)\sinh\lambda)\\ l = r(\cosh\lambda + \cosh(2u)\sinh\lambda) \end{cases}$$

needs to be studied in details to make the equations (2.29), (2.30) useful. We note first of all that, if (λ,u) is a solution of (3.1) corresponding to a given (f,l), then $(\lambda, -u)$ is also a solution. Thus $\mathcal{F}$ is certainly not a global diffeomorphism. This is confirmed by the form of the jacobian of (3.1)

$$J(\lambda, u; r) = r^2 \sinh(2u)(\cosh(2\lambda) - 1) \tag{3.3}$$

which vanishes if either $u = 0$ or $\lambda = 0$. Moreover, if (λ, u) is a solution of (3.1) corresponding to $(\tilde{f}, \tilde{l})$ then $(-\lambda, u)$ is a solution when $(f, l) = (\tilde{l}, \tilde{f})$. Thus the range $\mathcal{R}$ of $\mathcal{F}$ is symmetric with respect to the line $l = f$. Moreover, the range of $\mathcal{F}$ is strictly contained in $\mathbf{R}^2$. In fact, a complete description of $\mathcal{R}$ can be obtained adding and subtracting the first equation of (3.1) from the second one. We obtain, after simple calculation (see Figure 1),

$$\mathcal{R} = D_1 \cup D_2 \cup D_3 \cup D_4, \tag{3.4}$$

where

$$D_1 = \{(f, l);\ f + l > 2r,\ f > r,\ fl < r^2\} \tag{3.5}$$

$$D_2 = \{(f, l);\ f + l > 2r,\ 0 < f < r,\ fl < r^2\} \cup \{(f, l);\ f + l > 2r,\ f \leq 0\} \tag{3.6}$$

$$D_3 = \{(f, l);\ fl = r^2,\ f > \infty,\ f \neq r\} \tag{3.7}$$

$$D_4 = \{(f, l);\ f = r,\ l = r\}. \tag{3.8}$$

If $(f, l) \in D_1$ we have two solutions of (3.1), i.e.

(3.9)
$$\lambda = \log \Big[\frac{1}{2r}\Big(f + l - \sqrt{(f + l)^2 - 4r^2}\Big)\Big],\ u = \pm\frac{1}{2}\cosh^{-1}\Big(\frac{f - l}{\sqrt{(f + l)^2 - 4r^2}}\Big).$$

If $(f, l) \in D_2$ we have again two solutions of (3.1), i.e.

(3.10)
$$\lambda = \log \Big[\frac{1}{2r}\Big(f + l + \sqrt{(f + l)^2 - 4r^2}\Big)\Big],\ u = \pm\frac{1}{2}\cosh^{-1}\Big(\frac{l - f}{\sqrt{(f + l)^2 - 4r^2}}\Big).$$

If $(f, l) \in D_3$ we have the only solution $\lambda = \log \dfrac{f}{r}$, $u = 0$. Finally if $(f, l) \in D_4$ the system has infinite solutions given by $\lambda = 0$, $u = k$, $k \in \mathbf{R}^1$. We conclude that the reformulation in terms of λ and u of the system (2.22)-(2.25) with the system (2.29), (2.30) is acceptable only when (f, l) belongs to the range of $\mathcal{F}$.

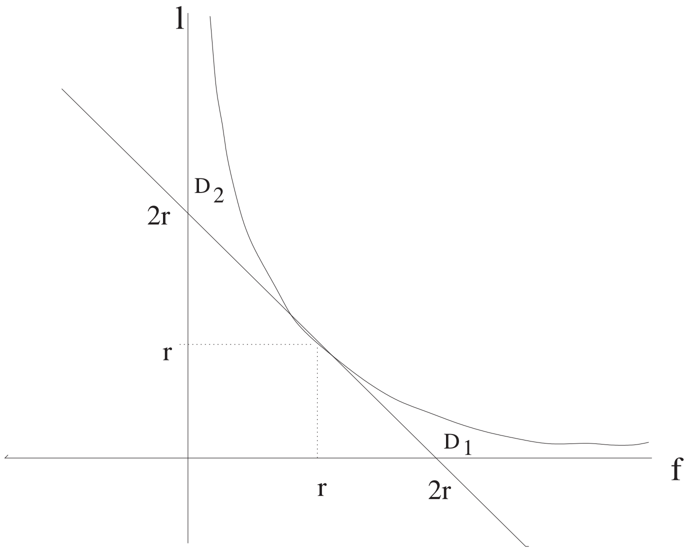

Fig. 1

4 - The functional solutions of system (2.29), (2.30) and the corresponding Bernoulli equation

The next Lemma refers to the system of P.D.E.

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\sigma(\lambda)\frac{\partial u}{\partial r}\Big)+\frac{\partial}{\partial z}\Big(\sigma(\lambda)\frac{\partial u}{\partial z}\Big)=0 \tag{4.1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial \lambda}{\partial r}\Big)+\frac{\partial^2 \lambda}{\partial z^2}+2\sigma'(\lambda)\Big[\Big(\frac{\partial u}{\partial r}\Big)^2+\Big(\frac{\partial u}{\partial z}\Big)^2\Big]=0 \tag{4.2}$$

which reduces to the system (2.29), (2.30) if $\sigma(\lambda)=\sinh^2\lambda$. The system (4.1), (4.2) can be more compactly written

$$\nabla\cdot(\sigma(\lambda)\nabla u)=0 \tag{4.3}$$

$$\nabla\cdot(\sigma(\lambda)\nabla u)+2\sigma'(\lambda)|\nabla u|^2=0, \tag{4.4}$$

where ∇ and $\nabla\cdot$ denote respectively the gradient and the divergence operators in cylindrical coordinates.

Definition 4.1. *We say that* $(\lambda(r,z),u(r,z))$ *is a functional solution of the system* (4.3), (4.4) *if there exists a regular function* $u=\mathcal{U}(\lambda)$ *such that*

$u(r,z) = \mathcal{U}(\lambda(r,z))$ or, alternatively, if there exists a function $\lambda = L(u)$ such that $\lambda(r,z) = L(u(r,z))$.

The ordinary differential equation which permits to find the function $\mathcal{U}(\lambda)$ or $L(u)$ of the above definition is given in the next

L e m m a 4.2. *Let $(\lambda(r,z), u(r,z))$ be a functional solution of (4.3), (4.4). If*

$$|\nabla\lambda(r,z)|^2 \neq 0, \ \ \text{and } \sigma(\lambda) \neq 0 \tag{4.5}$$

then the function $\mathcal{U}(\lambda)$ is a solution of the Bernoulli equation

$$\mathcal{U}'' + \frac{\sigma'(\lambda)}{\sigma(\lambda)}\mathcal{U}' - 2\sigma'(\lambda)\mathcal{U}'^3 = 0, \qquad \mathcal{U}' = \frac{d\mathcal{U}}{d\lambda}. \tag{4.6}$$

If

$$|\nabla u(r,z)|^2 \neq 0, \ \ \text{and } \sigma(\lambda) \neq 0 \tag{4.7}$$

then $L(u)$ is a solution of the autonomous equation

$$L'' - \frac{\sigma'(L)}{\sigma(L)}L'^2 + 2\sigma'(L) = 0, \qquad L' = \frac{dL}{du}. \tag{4.8}$$

P r o o f. Since $\nabla u = \mathcal{U}'(\lambda)\nabla\lambda$ the equations (4.3) and (4.4) become respectively

$$\nabla \cdot (\sigma(\lambda)\mathcal{U}'(\lambda)\nabla\lambda) = 0 \tag{4.9}$$

$$\nabla^2\lambda = -2\sigma'(\lambda)\mathcal{U}'^2(\lambda)|\nabla\lambda|^2. \tag{4.10}$$

From (4.9) we have

$$\sigma(\lambda)\mathcal{U}'(\lambda)\nabla^2\lambda + \sigma'(\lambda)\mathcal{U}'(\lambda)|\nabla\lambda|^2 + \sigma(\lambda)\mathcal{U}''(\lambda)|\nabla\lambda|^2 = 0. \tag{4.11}$$

Substituting (4.10) into (4.11) we have, by (4.5) and taking into account (4.10),

$$\mathcal{U}'' + \frac{\sigma'(\lambda)}{\sigma(\lambda)}\mathcal{U}' - 2\sigma'(\lambda)\mathcal{U}'^3 = 0.$$

The proof of (4.8) is the similar with minor changes. □

If $\sigma(\lambda) = \sinh^2\lambda$, as in the case of interest to us, (4.6) becomes the Bernoulli equation

$$\mathcal{U}'' + 2\mathcal{U}'\coth\lambda - 2\sinh(2\lambda)\mathcal{U}'^3 = 0. \tag{4.12}$$

If $\mathcal{U}(\lambda)$ is a solution of (4.12) also $-\mathcal{U}(\lambda)$ is a solution. Moreover, the general solution of (4.12) can be explicitly found. This is the main advantage of the present approach. We have, in addition to the trivial solution $\mathcal{U}(\lambda) = 0$, two families of solutions:

$$\mathcal{U}(\lambda) = T(H(c,\lambda)) + c_1 \tag{4.13}$$

and

$$\mathcal{U}(\lambda) = -T(H(c,\lambda)) - c_1, \tag{4.14}$$

where

$$T(x) = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right| \tag{4.15}$$

and

$$H(c,\lambda) = \frac{2(1+e^{2\lambda})}{\sqrt{c(e^{2\lambda}-1)^2 + 16e^{2\lambda}}}. \tag{4.16}$$

5 - A two-point problem for the Bernoulli equation (4.12)

In this section we study the problem

$$\mathcal{U}'' + 2\mathcal{U}'\coth\lambda - 2\sinh(2\lambda)\mathcal{U}'^3 = 0 \tag{5.1}$$

$$\mathcal{U}(a) = A,\ \mathcal{U}(b) = B, \tag{5.2}$$

where a, b, A, B are given constants satisfying

$$b > a > 0. \tag{5.3}$$

This two-point problem will be instrumental in dealing, in the last section of this paper, with a boundary value problem for the system (2.29), (2.30). In view of (4.13) and (4.14) the problem (5.1), (5.2) is reduced to the search of the solutions (c, c_1) of the system

$$T(H(c,a)) + c_1 = A \tag{5.4}$$

$$T(H(c,b)) + c_1 = B \tag{5.5}$$

and of the system

$$T(H(c,a)) + c_1 = -A \tag{5.6}$$

$$T(H(c,b)) + c_1 = -B. \tag{5.7}$$

We discuss (5.4), (5.5). By difference we have

$$F(c,a,b) = B - A, \tag{5.8}$$

where

$$F(c,a,b) = T(H(c,b)) - T(H(c,a)). \tag{5.9}$$

From (5.6), (5.7) we have, as an equally acceptable equation,

$$F(c,a,b) = A - B. \tag{5.10}$$

We study equation (5.8). To this end, we collect below certain properties of the function $H(c,\lambda)$ which is defined in the set

$$S = \{(c,\lambda);\ c > c^*(\lambda),\ \lambda \in \mathbf{R}^1\} = \{(c,\lambda);\ \lambda < \lambda^*(c),\ c \in \mathbf{R}^1\},$$

where $\lambda^*(c)$ is the inverse of $c^*(\lambda)$. We have

$$c^*(\lambda) = -\frac{16e^{2\lambda}}{(e^{2\lambda}-1)^2}, \quad \lambda > 0, \quad \lambda^*(c) = \frac{1}{2}\log\frac{c-8-4\sqrt{4-c}}{c}, \quad c < 0. \tag{5.11}$$

In view of (5.3) we are interested in the behaviour of $H(c,\lambda)$ in the smaller set

$$D = S \cap \{(c,\lambda);\ c > c^*(\lambda),\ \lambda \in \mathbf{R}^1\}.$$

We have

$$\lim_{c\to-\infty} \lambda^*(c) = 0, \quad \lim_{c\to 0-} \lambda^*(c) = \infty, \quad \lambda^{*'}(c) > 0$$

and

$$H(4,\lambda) = 1,\ H(0,\lambda) = \frac{1+e^{2\lambda}}{2e^{\lambda}},\ H(c,0) = 1. \tag{5.12}$$

Moreover, since

$$\frac{\partial H}{\partial \lambda}(c,\lambda) = \frac{16e^{3\lambda}(4-c)\sinh\lambda}{[16e^{2\lambda} + c(e^{2\lambda}-1)^2]^{3/2}} \tag{5.13}$$

we obtain

$$\frac{\partial H}{\partial \lambda}(c,\lambda) < 0 \text{ if } c > 4, \text{ and } \lambda > 0 \tag{5.14}$$

$$\frac{\partial H}{\partial \lambda}(c,\lambda) > 0 \text{ if } c^*(\lambda) < c < 4, \text{and } \lambda > 0. \tag{5.15}$$

If $0 < c < \infty$ we have

$$\lim_{\lambda\to\infty} H(c,\lambda) = 2c^{-1/2} \tag{5.16}$$

and, if $c^* < c < 0$,

$$\lim_{\lambda\to\lambda^*(c)^+} H(c,\lambda) = \infty.$$

In order to give a meaning to the function $F(c,a,b) = T(H(c,b)) - T(H(c,a))$ both $H(c,b)$ and $H(c,a)$ must be well-defined. Now, $H(c,b)$ is well-defined if $c > c^*(b)$ and $H(c,a)$ if $c > c^*(a)$. On the other hand, $c^*(b) > c^*(a)$ since $c^*(\lambda)$ is strictly increasing. Hence $F(c,a,b)$ is well-defined if

$$\infty > c > c^*(b). \tag{5.17}$$

We claim that $F(c,a,b) < 0$ in the range (5.17). We distinguish two cases. (i) If $4 < c < \infty$ we have

$$0 < H(c,b) < H(c,a) < 1 \tag{5.18}$$

but, in the interval $[0,1)$ the function $T(x)$ is strictly increasing. Therefore, by (5.18) we have $F(c,a,b) < 0$. (ii) If $c^*(b) < c < 4$ we have $H(c,b) > H(c,a) > 1$ with $T(x)$ strictly decreasing in $(1,\infty)$, hence, again, $F(c,a,b) < 0$.

The function $F(c,a,b)$ could, "a priori", have a singularity for $c = 4$. However, this is not the case. For, we have

$$\lim_{c\to 4^+} F(c,a,b) = \lim_{c\to 4^-} F(c,a,b) = \log\frac{(e^{2b}+1)(e^{2a}-1)}{(e^{2b}-1)(e^{2a}+1)}. \tag{5.19}$$

Hence $F(4,a,b) = \log\frac{(e^{2b}+1)(e^{2a}-1)}{(e^{2b}-1)(e^{2a}+1)} < 0$ because $\frac{(e^{2b}+1)(e^{2a}-1)}{(e^{2b}-1)(e^{2a}+1)} < 1$. Since

$$\lim_{c\to c^*(b)^+} T(H(c,b)) = T\big(\lim_{c\to c^*(b)^+} H(c,b)\big) = \lim_{x\to\infty} \frac{1}{2}\log\left|\frac{1+x}{1-x}\right| = 0,$$

we obtain

$$\lim_{c\to c^*(b)^+} F(c,a,b) = -T(H(c^*(b),a)) < 0.$$

Moreover, we have

$$\frac{\partial F}{\partial c}(c,a,b)=\frac{1}{c-4}\left[\frac{1+e^{2a}}{\sqrt{c(e^{2a}-1)^2+16e^{2a}}}-\frac{1+e^{2b}}{\sqrt{c(e^{2b}-1)^2+16e^{2b}}}\right].$$

This implies

$$\frac{\partial F}{\partial c}(c,a,b)=\frac{1}{2(c-4)}[H(c,a)-H(c,b)] \tag{5.20}$$

and the following

Lemma 5.1. *For* $c\in(c^*(b),\infty)$ *and* $b>a>0$, $\frac{\partial F}{\partial c}(c,a,b)$ *is real analytic and*

$$\frac{\partial F}{\partial c}(c,a,b)>0. \tag{5.21}$$

Proof. The only point in which $\frac{\partial F}{\partial c}$ could be singular is $c=4$. On the other hand, $H(c,a)$ and $H(c,b)$ are both real analytic and

$$H(4,a)=H(4,b). \tag{5.22}$$

As a consequence, the difference of the Taylor series of $H(c,a)$ and $H(c,b)$ with respect to c and with the same initial point $c=4$ gives, recalling (5.20),

$$\begin{aligned}\frac{\partial F}{\partial c}(c,a,b)=\frac{1}{2}\Big[\frac{\partial H}{\partial c}(4,a)-\frac{\partial H}{\partial b}(4,b)\Big]\\ +\frac{c-2}{4}\Big[\frac{\partial^2 H}{\partial c^2}(4,a)-\frac{\partial^2 H}{\partial c^2}(4,b)\Big]+\text{higher terms.}\end{aligned}$$

Thus $c=4$ is not singular and $\frac{\partial F}{\partial c}$ is real analytic. To prove (5.21) we define, for $c>c^*(b)$ and $b>a>0$,

$$G(c,a,b)=H(c,a)-H(c,b). \tag{5.23}$$

We claim that $G(c,a,b)$ vanishes if and only if $c=4$. This is equivalent to say that the equation in c

$$f(c)=\frac{(1+e^{2b})^2}{(1+e^{2a})^2},\quad\text{where}\quad f(c)=\frac{16e^{2b}+c(e^{2b}-1)^2}{16e^{2a}+c(e^{2a}-1)^2},$$

has the only solution $c=4$ and this fact is immediately seen since $f'(c)>0$ if $b>a>0$. On the other hand, it is easy to verify that $G(0,a,b)<0$

and $G(8, a, b) > 0$. Hence we have $G(c, a, b) < 0$ if $c < 4$, $G(4, a, b) = 0$ and $G(c, a, b) > 0$ if $c > 4$. Recalling that

$$\frac{\partial F}{\partial c}(c, a, b) = \frac{G(c, a, b)}{2(c - 4)}$$

we arrive at (5.21). □

By Lemma 5.1 we may conclude that the graphs of $F(c, a, b)$ and $-F(c, a, b)$ as functions of c are, for any $b > a > 0$, qualitatively, those of Figure 2.

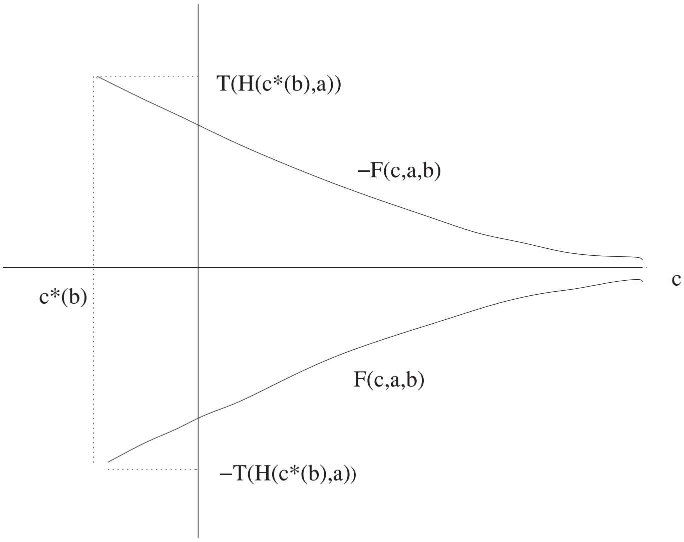

Fig. 2

We have:

Theorem 5.2. *Let us consider the two-point problem*

$$\frac{d^2\mathcal{U}}{d\lambda^2} + 2\frac{d\mathcal{U}}{d\lambda}\coth\lambda - \left(\frac{d\mathcal{U}}{d\lambda}\right)^3 \sinh(2\lambda) = 0 \tag{5.24}$$

$$\mathcal{U}(a) = A,\ \mathcal{U}(b) = B, \tag{5.25}$$

where a, b, A, B are given constants satisfying

$$b > a > 0. \tag{5.26}$$

If $B = A$ the problem (5.24), (5.25) *has only the trivial solution $\mathcal{U}(\lambda) = 0$. If $B < A$ and*

$$-T(H(c^*(b), a)) \leq (B - A) \tag{5.27}$$

the problem (5.24), (5.25) *has only and only one solution. If*

$$(B - A) < -T(H(c^*(b), a)) \tag{5.28}$$

the problem (5.24), (5.25) *has no solutions. If* $B > A$ *and*

$$(B - A) \leq T(H(c^*(b), a)) \tag{5.29}$$

the problem (5.24), (5.25) *has only and only one solution. If*

$$(B - A) > T(H(c^*(b), a)) \tag{5.30}$$

the problem (5.24), (5.25) *has no solutions.*

Proof. If $B < A$ the solutions of (5.24), (5.25) are in a one-to-one correspondence with the solutions of the equation

$$F(c, a, b) = (B - A), \tag{5.31}$$

where

$$F(c, a, b) = T(H(c, b)) - T(H(c, a)).$$

Assume (5.27). By Lemma 5.1 the equation (5.31) has one and only one solution $\tilde{c}$ (see Figure 2). From (5.4) we have

$$\tilde{c}_1 = -T(H(\tilde{c}, a)) + A. \tag{5.32}$$

Thus the unique solution of problem (5.24), (5.25) is given, in this case, by

$$\mathcal{U}(\lambda) = T(H(\tilde{c}, \lambda)) - T(H(\tilde{c}, a)) + A. \tag{5.33}$$

If, on the other hand, we assume (5.28) the equation (5.31) has no solution and therefore this is also the case for problem (5.24), (5.25). The two remaining cases can be proved in the same way starting now with equation $-F(c, a, b) = (B - A)$. □

6 - Existence and non-existence for problem (6.1)-(6.4)

In this section we use Theorem 5.2 to prove a result of existence, uniqueness and non-existence of functional solutions for a boundary value problem for the system of Einstein's equations

$$\frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{r}\frac{\partial f}{\partial r} = -\frac{f}{r^2}([f, l] + [m, m]) \tag{6.1}$$

$$\frac{\partial^2 l}{\partial r^2} + \frac{\partial^2 l}{\partial z^2} - \frac{1}{r}\frac{\partial l}{\partial r} = -\frac{l}{r^2}([f,l] + [m,m]) \tag{6.2}$$

$$\frac{\partial^2 m}{\partial r^2} + \frac{\partial^2 m}{\partial z^2} - \frac{1}{r}\frac{\partial m}{\partial r} = -\frac{m}{r^2}([f,l] + [m,m]) \tag{6.3}$$

$$fl + m^2 = r^2. \tag{6.4}$$

All the matter producing the field is contained inside an indefinite circular cylinder of surface Γ_1 and radius 1 with axis coincident with the z axis. This matter is supposed to determine on Γ_1 the values of f and l as given constants f_1 and l_1, i.e.

$$f(1,z) = f_1,\ \ l(1,z) = l_1. \tag{6.5}$$

A second cylindrical surface Γ_R, coaxial to Γ_1 and of radius $R > 1$, is taken as "horizon" ([3]). We assume on Γ_R the values of f and l pertaining to the flat space solution, i.e.

$$f(R,z) = 1,\ \ l(R,z) = R^2. \tag{6.6}$$

Moreover, we assume

$$\lim_{z\to\pm\infty} \frac{\partial f}{\partial z}(r,z) = 0,\ \ \lim_{z\to\pm\infty} \frac{\partial l}{\partial z}(r,z) = 0. \tag{6.7}$$

These conditions at infinity are compatible with the request of flatness at infinity. We wish to solve the system (6.1)-(6.4) in the region Ω between the two surfaces Γ_1, Γ_R with the boundary conditions (6.5)-(6.7). To this end we reformulate this boundary value problem in terms of $\lambda(r,z)$ and $u(r,z)$ with the aid of the transformation (3.1). This will lead to the simpler system

$$\nabla \cdot (\sinh^2 \lambda \nabla u) = 0 \tag{6.8}$$

$$\nabla^2 \lambda + 2\sinh(2\lambda)|\nabla u|^2 = 0. \tag{6.9}$$

With this approach, however, care must be taken to accept only boundary values for f and l which belong to the range of (3.1) to make the formulation meaningful. For the boundary condition (6.6) there is no problem since $(1, R^2)$ belongs to the range of (3.1) if $r = R$. For the boundary condition (6.6), on the contrary, we must assume (f_1, l_1) to belong to the range of (3.1) if $r = 1$.

[3] If R is very large in comparison with 1 this does not seem to be too unrealistic.

This means to suppose, for example, $(f_1, l_1) \in D_2$ i.e., in view of the results of Section 3, ([4])

$$f_1 l_1 < 1,\ \ f_1 + l_1 > 2,\ \ f_1 < 1. \tag{6.10}$$

To find the boundary conditions corresponding to (6.6) in terms of λ and u we need to solve with respect to (λ, u) the system

$$\begin{cases} R(e^{-\lambda}\cosh^2 u - e^{\lambda}\sinh^2 u) = 1 \\ R(e^{\lambda}\cosh^2 u - e^{-\lambda}\sinh^2 u) = R^2. \end{cases} \tag{6.11}$$

Since $R > 1$, it is easy to see that the only solution of (6.11) is given by

$$(\lambda, u) = (\log R, 0). \tag{6.12}$$

The boundary conditions corresponding to (6.5) in terms of λ and u are obtained solving the system

$$\begin{cases} e^{-\lambda}\cosh^2 u - e^{\lambda}\sinh^2 u = f_1 \\ e^{\lambda}\cosh^2 u - e^{-\lambda}\sinh^2 u = l_1. \end{cases} \tag{6.13}$$

We must distinguish various cases according to the position of (f_1, l_1) in the range of (3.1). If we assume e.g. (6.10) we have, in view of (3.10) two solutions $(\lambda, u) = (a, A)$ and $(\lambda, u) = (a, -A)$ with $a > 0$ and $A > 0$ ([5]). We proceed with the first solution, i.e. (a, A). There is no indeterminacy in this choice, since with both solutions we arrive at the same (f_1, l_1) which is the physically significant datum. We obtain the boundary value problem

$$\nabla \cdot (\sinh^2 \lambda \nabla u) = 0 \ \text{ in } \Omega \tag{6.14}$$

$$\nabla^2 \lambda + 2\sinh(2\lambda)|\nabla u|^2 = 0 \ \text{ in } \Omega \tag{6.15}$$

$$\lambda(1, z) = a,\ \ u(1, z) = A \tag{6.16}$$

$$\lambda(R, z) = \log R,\ \ u(R, z) = 0 \tag{6.17}$$

$$\lim_{z \to \pm\infty} \frac{\partial \lambda}{\partial z}(r, z) = 0 \tag{6.18}$$

[4]The other cases, in which (f_1, l_1) belongs to other parts of the range of $\mathcal{F}$, can be dealt with in a similar manner.

[5]According to (3.10) $a = \log\left[\frac{1}{2}\left(f + l + \sqrt{(f+l)^2 - 4}\right)\right]$ and $A = \frac{1}{2}\cosh^{-1}\left(\frac{l-f}{\sqrt{(f+l)^2-4}}\right)$

$$\lim_{z\to\pm\infty}\frac{\partial u}{\partial z}(r,z)=0. \tag{6.19}$$

We search the functional solutions of problem (6.14)-(6.19) in the sense of Section 4. Therefore we consider the corresponding Bernoulli equation i.e

$$\frac{d^2\mathcal{U}}{d\lambda^2}+2\frac{d\mathcal{U}}{d\lambda}\coth\lambda-\Big(\frac{d\mathcal{U}}{d\lambda}\Big)^3\sinh(2\lambda)=0. \tag{6.20}$$

In view of the boundary conditions (6.16), (6.17) we add to (6.20) the conditions

$$\mathcal{U}(a)=A,\ \mathcal{U}(\log R)=0. \tag{6.21}$$

If we assume $\log R > a$ ([6]) we can apply Theorem 5.2 to the two-point problem (6.20), (6.21) with $b=\log R$ and $B=0$. Thus, if

$$T(H(c^*(\log R),a))\geq A \tag{6.22}$$

the two-point problem (6.20), (6.21) has one and only one solution and when

$$A>T(H(c^*(\log R),a)) \tag{6.23}$$

it has no solution. Let $(\lambda(r,z),u(r,z))$ be a functional solution of the boundary value problem (6.14)-(6.19). We have $u(r,z)=\mathcal{U}(\lambda(r,z))$ with $\mathcal{U}(\lambda)$ a solution of (6.20), (6.21), therefore

$$\nabla u=\mathcal{U}'(\lambda)\nabla\lambda. \tag{6.24}$$

The equation (6.14) becomes

$$\nabla\cdot(S(\lambda)\nabla\lambda)=0\ \text{ in }\Omega, \tag{6.25}$$

where

$$S(\lambda)=\mathcal{U}'(\lambda)\sinh^2\lambda. \tag{6.26}$$

To (6.25) we must add the boundary conditions

$$\lambda(1,z)=a \tag{6.27}$$

$$\lambda(R,z)=\log R \tag{6.28}$$

$$\lim_{z\to\pm\infty}\frac{\partial\lambda}{\partial z}(r,z)=0. \tag{6.29}$$

[6] The external cylinder Γ_R is our "horizon". Therefore, this hypothesis is not restrictive.

The nonlinear Dirichlet's problem (6.25)-(6.29) can be solved with the aid of the Kirchhoff's transformation. Let us define

$$w = G(\lambda), \quad \text{where} \quad G(\lambda) = \int_a^\lambda S(t)dt. \tag{6.30}$$

In term of w the problem (6.25), (6.27), (6.28) and (6.29) becomes

$$\nabla^2 w = 0 \ \text{ in } \Omega \tag{6.31}$$

$$w(0, z) = 0 \tag{6.32}$$

$$w(\log R, z) = G(\log R) \tag{6.33}$$

$$\lim_{z\to\pm\infty} \frac{\partial w}{\partial z}(r, z) = 0. \tag{6.34}$$

The solution of problem (6.31)-(6.34) is

$$w(r) = \frac{G(\log R)\log r}{\log R}. \tag{6.35}$$

From (6.30) we have

$$G(\lambda) = \frac{G(\log R)\log r}{\log R}. \tag{6.36}$$

The function $S(\lambda) = \dfrac{dG}{d\lambda}(\lambda)$ has a positive lower bound if $\lambda \in [a, \log R]$. Hence $G(\lambda)$ is globally invertible. Thus we can solve (6.36) with respect to λ obtaining

$$\tilde{\lambda}(r) = G^{-1}\Big(\frac{G(\log R)\log r}{\log R}\Big). \tag{6.37}$$

Hence, as functional solution of problem (6.14)-(6.19), we have

$$(\lambda, u) = (\tilde{\lambda}(r), \mathcal{U}(\tilde{\lambda}(r))). \tag{6.38}$$

Using the parametrization (3.2) we obtain the solution of the boundary value problem (6.1)-(6.6)

$$\begin{cases} f(r,z) &= r(\cosh(\tilde{\lambda}(r)) - \cosh(2\mathcal{U}(\tilde{\lambda}(r)))\sinh(\tilde{\lambda}(r))) \\ l(r,z) &= r(\cosh(\tilde{\lambda}(r)) + \cosh(2\mathcal{U}(\tilde{\lambda}(r)))\sinh(\tilde{\lambda}(r))) \\ m(r,z) &= r\sinh(\tilde{\lambda}(r))\sinh(2\mathcal{U}(\tilde{\lambda}(r))). \end{cases} \tag{6.39}$$

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