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The b-secant variety of a smooth curve has a codimension 1 locally closed subset whose points have rank at least $b+1$

Abstract. Take a smooth, connected and non-degenerate projective curve $X \subset \mathbb{P}^r$, $r \geq 2b+2 \geq 6$, defined over an algebraically closed field with characteristic 0 and let $\sigma_b(X)$ be the b-secant variety of X. We prove that the X-rank of q is at least $b+1$ for a non-empty codimension 1 locally closed subset of $\sigma_b(X)$.

Keywords. Secant variety, X-rank, tangential variety, join of two varieties, tangentially degenerate curve, strange curve

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1 - Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate projective variety defined over an algebraically closed field. For any $q \in X$ the X-rank $r_X(q)$ of X is the minimal cardinality of a set $S \subset X$ such that $q \in \langle S \rangle$, where $\langle \ \rangle$ denotes the linear span. For any integer $s > 0$ let $\sigma_s(X) \subseteq \mathbb{P}^r$ be the s-secant variety of X, i.e. the closure of the union of all linear spaces $\langle S \rangle$ with $S \subset X$ and $\sharp(S) = s$. See [**12**] for many applications of X-ranks (e.g. the tensor rank) and secant varieties (a.k.a. the border rank). The algebraic set $\sigma_s(X)$ is an integral projective variety of dimension $\leq s(1+\dim X)-1$ and $\sigma_s(X)$ is said to be non-defective if it has dimension $\min\{r, s(1+\dim X)-1\}$. Every secant variety of a curve is non-defective ([**1**, Corollary 1.4]). Let $\tau(X) \subseteq \mathbb{P}^r$ be the tangential variety of X, i.e. the closure in $\mathbb{P}^r$ of the union of all tangent spaces

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T_pX, $p \in X_{\mathrm{reg}}$. The algebraic set $\tau(X)$ is an integral projective variety of dimension $\le 2(\dim X)$ and $\tau(X) \subseteq \sigma_2(X)$. For any integer $b \ge 2$ let $\tau(X, b)$ denote the join of one copy of $\tau(X)$ and $b-2$ copies of X. If X is a curve, then $\dim \tau(X, b) = \min\{r, 2b-2\}$ (use $b-2$ times [**1**, part 2) of Proposition 1.3] and that $\dim \tau(X) = 2$) and hence $\tau(X, b)$ is a non-empty codimension 1 subset of $\sigma_b(X)$ if X is a curve and $r > 2b$. For a projective variety X of arbitrary dimensional usually $\tau(X, b)$ is a hypersurface of $\sigma_b(X)$, but this is not always true. For instance, if $\sigma_b(X)$ has not the expected dimension one expects that $\tau(X, b) = \sigma_b(X)$ and this is the case if X is smooth ([**6**, Corollary 4]).

Question 1.1. *Assume $b \ge 2$, $r \ge b(1 + \dim X) - 2$, and that $\sigma_b(X)$ has the expected dimension. Is $r_X(q) > b$ for a non-empty locally closed subset of $\sigma_b(X)$ of codimension 1 in $\sigma_b(X)$? Is $r_X(q) > b$ for a general point of $\tau(X, b)$?*

In this note we prove the following result.

Theorem 1.2. *Fix an integer $b \ge 2$ and let $X \subset \mathbb{P}^r$, $r \ge 2b+2$, be an smooth, connected and non-degenerate projective curve defined over an algebraically closed field with characteristic 0. Let q be a general element of $\tau(X, b)$. Then $r_X(q) > b$.*

From Theorem 1.2 we easily get the following result.

Corollary 1.3. *Take b and X as in Theorem 1.2. Then there is a quasi-projective variety $J \subset \sigma_b(X)$ such that $\dim J = \dim \sigma_b(X) - 1$ and $r_X(q) > b$ for all $q \in J$.*

Let $X \subset \mathbb{P}^r$, $r \ge 3$, be an integral and non-degenerate projective curve. X is said to be *tangentially degenerate* if a general tangent line of X_{reg} meets X at another point of X. H. Kaji proved that in characteristic 0 a non-degenerate smooth projective curve or a projective curve for which the normalization map $C \to X \subset \mathbb{P}^r$ is unramified is not tangentially degenerate ([**10**, Theorem 3.1 and Remark 3.8]). M. Bolognesi and G. Pirola extended this result to curves with toric singularities ([**4**]). A. Terracini gave an example of a tangentially degenerate analytic curve in $\mathbb{C}^3$ ([**17**, page 143]). In positive characteristic there are many examples of non-strange curves, which are tangentially degenerate ([**10**, Examples 4.1 and 4.2], [**5**, §5], [**7**, Example 3], [**14**, Example at page 137]). See [**11**] for further results on this topic.

If for a general $p \in X_{\mathrm{reg}}$ the tangent space T_pX is the 2-secant variety of the reduced projective set $(X \cap T_pX)_{\mathrm{red}}$, then a general $q \in \tau(X)$ has X-rank 2 and hence for every integer $b \ge 2$ the X-rank of a general element of $\tau(X, b)$ is at most b.

In Question 1.1 we exclude the case $r = b(1 + \dim X) - 1$, because in this case usually the answer would be NO (see e.g. [**15**] for the case of space curves). Usually NO, but in a few cases YES, as for instance when $r = 2b - 1$ and X is a rational normal curve by a theorem of Sylvester's ([**2**, Theorem 23], [**3**], [**9**, §1.3], [**13**, §4]).

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2 - The proof of Theorem 1.2

We work over an algebraically closed field with characteristic 0. Our main reason to require in Theorem 1.2 both characteristic zero and that the curve is smooth is that [**8**] requires both assumptions. Of course, in positive characteristic and/or for singular curves we also need to assume (at the very least) that X is not tangentially degenerate (Remark 2.3).

Let X be an integral projective curve defined over an algebraically closed field with characteristic 0. For any integer $b \ge 2$ let $\mathcal{Z}(X, b)$ denote the set of all zero-dimensional schemes $Z \subset X$ with $\deg(Z) = b$, $b-1$ connected components and with the degree 2 connected component, v, of Z with $v_{\text{red}} \in X_{\text{reg}}$ (any such v is called a tangent vector of X_{reg}).

Lemma 2.1. *Let $X \subset \mathbb{P}^r$, $r \ge 4$, be an integral and non-degenerate projective curve. For any $o \in X$ let $X[o] \subset \mathbb{P}^{r-1}$ denote the closure of $\ell_o(X \setminus \{o\})$ in $\mathbb{P}^{r-1}$, where $\ell_o : \mathbb{P}^r \setminus \{o\} \to \mathbb{P}^{r-1}$ is the linear projection from o. We have $r_X(q) > 2$ for a general $q \in \tau(X)$ if $X[a]$ is not tangentially degenerate for a general $a \in X$.*

Proof. Since $X[a]$ is not tangentially degenerate for a general $a \in X$, X is not tangentially degenerate. Take a general $v \in \mathcal{Z}(X, 2)$ and set $\{p\} := v_{\text{red}}$ and $L := \langle v \rangle$. Assume that for a general $o \in L$ there is $S_o \subset X$ with $\sharp(S_o) \le 2$ and $o \in \langle S_o \rangle$. Since $L \cap X$ is finite, we have $\sharp(S_o) = 2$ for a general $o \in L$. Write $S_o = \{p_1(o), p_2(o)\}$. Since X is not tangentially degenerate, we have $L \ne \langle S_o \rangle$. For a general $o \in L$ the point $p_1(o)$ is general in X. Hence $X[p_1(o)]$ is a general inner projection of X. More precisely, for a general (p, o) the pair $(p, p_1(o))$ is general in X^2 and in particular $p \ne p_1(o)$. Since $(p, p_1(o))$ is general in X^2, $\ell_{p_1(o)}(p)$ is a general point point of $X[p_1(o)]$ and in particular $X[p_1(o)]$ is smooth at $\ell_{p_1(o)}(p)$. By construction $\ell_{p_1(o)}(p_2(o))$ is contained in the tangent line of $X[p_1(o)]$ at $\ell_{p_1(o)}(p)$. Since $\langle S_o \rangle \ne L$, we have $\ell_{p_1(o)}(p_2(o)) \ne \ell_{p_1(o)}(p)$ and so $X[p_1(o)]$ is tangentially degenerate, a contradiction. □

Remark 2.2. Let $Y \subset \mathbb{P}^m$, $m \ge 4$, be an integral and non-degenerate projective curve. Fix an integer s such that $1 \le s \le m - 3$ and a general

$(p_1, \dots, p_s) \in Y^s$. Set $V := \langle \{p_1, \dots, p_s\} \rangle$. Since Y is non-degenerate, we have $\dim V = s - 1$. The trisecant lemma implies that $Y \cap V = \{p_1, \dots, p_s\}$ (as schemes) and that the linear projection $\ell_V : \mathbb{P}^m \setminus V \to \mathbb{P}^{m-s}$ maps $Y \setminus Y \cap V$ birationally onto its image.

Remark 2.3. Assume $r \geq 2b \geq 6$ and fix a general $q \in \tau(X, b)$ and a general $o \in \tau(X)$. We claim that $r_X(q) \leq b - 2 + r_X(o)$. Indeed, by the definition of join we have $q \in \langle Z \rangle$, where Z is a general element of $\mathcal{Z}(X, b)$. Write $Z = v \cup \{p_1, \dots, p_{b-2}\}$ with $(v, p_1, \dots, p_{b-2})$ a general element of $\mathcal{Z}(X, 2) \times X^{b-2}$. Set $V := \langle Z \rangle$, $L := \langle v \rangle$, and $\{p\} := v_{\mathrm{red}}$. Since v is general in $\mathcal{Z}(X, 2)$, a general element of $\langle v \rangle$ is general in $\tau(X)$ and hence it has rank $r_X(o)$. Thus q has rank at most $r_X(o) + b - 2$. Thus Theorem 1.2 cannot be extended to tangentially degenerate curves.

Lemma 2.4. *Let $C \subset \mathbb{P}^r$, $r \geq 4$, be a smooth, connected and non-degenerate projective curve. Fix a general $(p_1, p_2) \in C^2$ and let $v = 2p_1$ denote the degree 2 connected effective divisor of C with p_1 as its reduction. Then $\dim \langle v \cup \{p_2\} \rangle = 2$ and $C \cap \langle v \cup \{p_2\} \rangle = v \cup \{p_2\}$ as schemes.*

Proof. We have $\dim \langle v \cup \{p_2\} \rangle = 2$, because C is non-degenerate. Let $3p_1 \subset C$ be the degree 3 effective connected divisor with p_1 as its support. Since we are in characteristic 0, a general point of C is not a hyperosculating point and so $\dim \langle 3p_1 \rangle = 2$ and $\langle 3p_1 \rangle \cap C$ does not contains p_1 with multiplicity > 3. We degenerate $v \cup \{p_2\}$ to the effective divisor $3p_1$ and apply [**8**, Theorem 1.9] to $3p_1$. □

Lemma 2.5. *Let $X \subseteq \mathbb{P}^r$,$r \geq 4$, be a smooth, irreducible and non-degenerate projective curve. For any $o \in X$ let $\ell_o : \mathbb{P}^r \setminus \{o\} \to \mathbb{P}^{r-1}$ be the linear projection from o. Call $X[o]$ the closure of $\ell_o(X \setminus \{o\})$ in $\mathbb{P}^{r-1}$. Then:*

1. *X is not tangentially degenerate and for a general $o \in X$ the curve $X[o]$ is not tangentially degenerate.*
2. *Let $L \subset \mathbb{P}^r$ be the tangent line of X at a general point of X. Let $\ell_L : \mathbb{P}^r \setminus L \to \mathbb{P}^{r-2}$ denote the linear projection from L. Then $\ell_{L|X \setminus X \cap L}$ is birational onto its image.*
3. *$r_X(q) > 2$ for a general $q \in \tau(X)$.*

Proof. X is not tangentially degenerate by [**10**, Theorem 3.1]. Fix a general $(o, p) \in X^2$ and set $p' = \ell_o(p)$. We have $\ell_o(T_pX) = T_{p'}X[o]$. The set $\Sigma := X[o] \setminus \ell_o(X \setminus \{o\})$ is finite (it is a single point, the point $\ell_o(T_oX \setminus \{o\})$, but we only need that it is finite). Assume the existence of $q \in X[o]$ with $q \neq p'$

and $q \in T_{p'}X[o]$. Since we are in characteristic zero, $X[o]$ is not strange. Hence for a general $p' \in X[o]$ we may assume that $T_{p'}X[o] \cap \Sigma = \emptyset$. Thus there is $q' \in X \setminus \{o\}$ with $\ell_o(q') = q$. By construction $\langle \{o, q'\} \cup T_pX \rangle$ is a plane. Note that (p, o) are general in X^2. Thus the existence of q' contradicts Lemma 2.4.

Part (3) follows from the second assertion of part (1) and Lemma 2.1.

Now we prove part (2). Assume that $\ell_{L|X\setminus L\cap X}$ is not birational onto its image and call $x \geq 2$ its degree. Fix a general $q \in X$. The plane $\langle \{q, L\} \rangle$ contains $x - 1$ other points of X, contradicting Lemma 2.4. $\square$

Lemma 2.6. *Fix an integer $b \geq 2$. Let $X \subset \mathbb{P}^r$, $r \geq 2b$, be a smooth and connected projective variety. Take a general $Z \in \mathcal{Z}(X, b)$ and set $V := \langle Z \rangle$. Then $\dim V = b - 1$ and the linear projection $\ell_V : \mathbb{P}^r \setminus V \to \mathbb{P}^{r-b}$ induces a birational map of X.*

Proof. Since X is non-degenerate, we have $\dim V = b - 1$. Write $Z = v \cup \{p_1, \dots, p_{b-2}\}$ with v connected of degree 2. Set $L := \langle v \rangle$. If $b = 2$, the lemma is part (2) of Lemma 2.5. Now assume $b > 2$. Let $\ell_L : \mathbb{P}^r \setminus L \to \mathbb{P}^{r-2}$ denote the linear projection from L and $Y \subset \mathbb{P}^{r-2}$ the closure of the image of $\ell_L(X \setminus X \cap L)$ in $\mathbb{P}^{r-2}$. Since Z is general, $p_i \notin L$ for all i and $(\ell_L(p_1), \dots, \ell_L(p_{b-2}))$ is general in Y^{b-2}. Apply Remark 2.2 to Y. $\square$

Proof [Proof of Theorem 1.2]. Since the case $b = 2$ is true by part (3) of Lemma 2.5, we may assume $b > 2$ and use induction on b. Fix a general $q \in \tau(X, b)$. Since $\dim \sigma_{b-1}(X) < \dim \tau(X, b)$, we have $r_X(q) \geq b$. By the definition of join we have $q \in \langle Z \rangle$, where Z is a general element of $\mathcal{Z}(X, b)$. Write $Z = v \cup \{p_1, \dots, p_{b-2}\}$ with $(v, p_1, \dots, p_{b-2})$ a general element of $\mathcal{Z}(X, 2) \times X^{b-2}$. Set $V := \langle Z \rangle$, $L := \langle v \rangle$, and $\{p\} := v_{\mathrm{red}}$. Since v is general in $\mathcal{Z}(X, 2)$, a general element of $\langle v \rangle$ is general in $\tau(X)$ and hence it has rank > 2. Since $p_1, \dots, p_{b-2}$ are general, we have $\dim V = b - 1$. Assume that for a general $q \in V$ there is a finite set $S_q \subset X$ with $\sharp(S_q) = b$ and $q \in \langle S_q \rangle$. Since $r_X(q) \geq b$, S_q is linearly independent. Set $L := \langle v \rangle$ and $\{p\} := v_{\mathrm{red}}$. Since X is not tangentially degenerate ([**10**, Theorem 3.1]) and p is general, we have $(X \cap L)_{\mathrm{red}} = \{p\}$. Since a general osculating space of X is not a hyperosculating one, we have $X \cap L = v$ as schemes. Since $q \in V \cap \langle S_q \rangle$, we have $\rho := \dim V \cap \langle S_q \rangle \geq 0$. Let $\ell_V : \mathbb{P}^r \setminus V \to \mathbb{P}^{r-b}$ denote the linear projection from V. Let $W \subset \mathbb{P}^{r-b}$ be the closure of $\ell_V(X \setminus X \cap V)$ in $\mathbb{P}^{r-b}$.

Claim 1. $(V \cap X)_{\mathrm{red}} = \{p, p_1, \dots, p_{b-2}\}$ and $V \cap X = Z$ (as schemes).

Proof of Claim 1. By Lemma 2.6 $\ell_{V|X\setminus X\cap V}$ is birational onto its image. Since $(p_1, \dots, p_{b-2})$ is general in X^{b-2}, we have $L \cap \{p_1, \dots, p_{b-2}\} = \emptyset$ and $(\ell_V(p_1), \dots, \ell_V(p_{b-2}))$ is a general element of W^{b-2}. By Remark 2.2, the fact that $X \cap L = v$ as schemes and the birationality of $\ell_{V|X\setminus X\cap V}$ we have $V \cap X = Z$ (as schemes).

Claim 2. $V \nsubseteq \langle S_q\rangle$.

Proof of Claim 2. We have $\dim\langle S_q\rangle = \dim V$ and $\sharp(V \cap X)_{\mathrm{red}} = b-1$ by Claim 1.

By Claim 2 we have $\rho \le b-2$. Any covering family of ρ-dimensional linear subspaces of V has dimension at least $b-1-\rho$. Thus any family $\{S_q\}_{q\in T}$ with $\cup_{q\in T}(V \cap \langle S_q\rangle\})$ covering a dense subset of V has dimension at least $b-1-\rho$.

(a) Assume $S_q \cap \{p, p_1, \dots, p_{b-2}\} = \emptyset$. We have $\dim\langle\ell_V(S_b)\rangle = b-2-\rho$. Taking a ramified covering of the parameter space T we may take $S'_q \subset S_q$ with $\sharp(S'_q) = b-1-\rho$ and $\ell_V(S'_q)$ linearly independent. Thus a general $A \in W^{b-1-\rho}$ has the property that $\langle A\rangle$ contains $1+\rho$ points of W (if $A = \ell_V(S_q)$ for a general $q \in T$, use the set $\ell_V(S_q \setminus S'_q)$). This is false by Remark 2.2.

(b) Assume $S_q \cap \{p_1, \dots, p_{b-2}\} \neq \emptyset$, say $p_{b-2} \in S_q$. Since $q \neq p_{b-2}$, $\langle v \cup \{p_1, \dots, p_{b-3}\}\rangle \cap \langle S_q \setminus \{p_{b-2}\}\rangle \neq \emptyset$. Fix a general $q' \in \langle v \cup \{p_1, \dots, p_{b-3}\}\rangle \cap \langle S_q \setminus \{p_{b-2}\}\rangle$. Since q is general in V, q' is general in $\langle v \cup \{p_1, \dots, p_{b-3}\}\rangle$. The set $S_q \setminus \{p_{b-2}\}$ shows that $r_X(q') \le b-1$, contradicting the inductive assumption.

(c) Assume $p \in S_q$. Since $q \neq p$, we have $\rho := \dim V \cap \langle S_q\rangle > 0$. By step (b) we may assume that $S_q \cap \{p_1, \dots, p_{b-2}\} = \emptyset$. Hence $\ell_V(S_q \setminus \{p\})$ is a well-defined subset of W with cardinality $b-1$ and spanning a linear subspace of dimension $b-2-\rho$. We take $A_q \subset S_q \setminus \{p\}$ with cardinality $b-1-\rho$ and with $\langle\ell_V(A_q)\rangle = \langle\ell_V(S_q \setminus \{p\})\rangle$. As in step (a) we may view $\ell_V(A_q)$ as a general subset of W with cardinality $b-1-\rho$, while its linear span contains $\ell_V(S_q \setminus \{p\})$, contradicting Remark 2.2. □

Remark 2.7. Assume characteristic 0. If no integral and non-degenerate curve in $\mathbb{P}^r$ is tangentially degenerate, then the proofs just given show that Theorem 1.2 holds also for singular curves.

References

[**1**] B. Ådlandsvik, *Joins and higher secant varieties*, Math. Scand. **62** (1987), 213–222.

[**2**] A. Bernardi, A. Gimigliano and M. Idà, *Computing symmetric rank for symmetric tensors*, J. Symbolic Comput. **46** (2011), 34–53.

[**3**] G. Comas and M. Seiguer, *On the rank of a binary form*, Found. Comput. Math. **11** (2011), 65–78.

[**4**] M. Bolognesi and G. Pirola, *Osculating spaces and Diophantine equations*, with an Appendix by P. Corvaja and U. Zannier, Math. Nachr. **284** (2011), 960–972.

[**5**] E. Esteves and M. Homma, *Order sequences and rational curves*, in: "Projective geometry with applications", Lecture Notes in Pure and Appl.Math., **166**, Dekker, New York, 1994, 27–42.

[**6**] W. Fulton and J. Hansen, *A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings*, Ann. of Math. **110** (1979), 159–166.

[**7**] A. Garcia and J. F. Voloch, *Duality for projective curves*, Bol. Soc. Brasil. Mat. (N.S.) **21** (1991), 159–175.

[**8**] S. González and R. Mallavibarrena, *Osculating degeneration of curves*, Special issue in honor of Steven L. Kleiman, Comm. Algebra **31** (2003), 3829–3845.

[**9**] A. Iarrobino and V. Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, Appendix C by Iarrobino and Steven L. Kleiman, Lecture Notes in Math., **1721**, Springer-Verlag, Berlin, 1999.

[**10**] H. Kaji, *On the tangentially degenerate curves*, J. London Math. Soc. (2) **33** (1986), 430–440.

[**11**] H. Kaji, *On the tangentially degenerate curves, II*, Bull. Braz. Math. Soc. (N.S.) **45** (2014), 745–752.

[**12**] J. M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics, **128**, Amer. Math. Soc., Providence, 2012.

[**13**] J. M. Landsberg and Z. Teitler, *On the ranks and border ranks of symmetric tensors*, Found. Comput. Math. **10** (2010), 339–366.

[**14**] D. Levcovitz, *Bounds for the number of fixed points of automorphisms of curves*, Proc. London Math. Soc. (3) **62** (1991), 133–150.

[**15**] R. Piene, *Cuspidal projections of space curves*, Math. Ann. **256** (1981), 95–119.

[**16**] J. Rathmann, *The uniform position principle for curves in characteristic p*, Math. Ann. **276** (1987), 565–579.

[**17**] A. Terracini, *Sulla riducibilità di alcune particolari corrispondenze algebriche*, Rend. Circ. Mat. Palermo **56** (1932), 112–143.

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