Riv. Mat. Univ. Parma, Vol. **8** (2017), 353-366

Mattia Brescia and Francesco de Giovanni

# Groups satisfying the double chain condition on non-pronormal subgroups

**Abstract.** If $\theta$ is a subgroup property, a group $G$ is said to satisfy the double chain condition on $\theta$-subgroups if it admits no infinite double sequences

$$\dots < X_{-n} < \dots < X_{-1} < X_0 < X_1 < \dots < X_n < \dots$$

consisting of $\theta$-subgroups. The structure of generalized soluble groups satisfying the double chain condition on non-pronormal subgroups is investigated.

**Keywords.** Double chain condition, pronormal subgroup, $T$-group.

**Mathematics Subject Classification (2010):** 20E15.

## 1 - Introduction

A subgroup $X$ of a group $G$ is said to be *pronormal* if the subgroups $X$ and $X^g$ are conjugate in $\langle X, X^g \rangle$ for every element $g$ of $G$. Obvious examples of pronormal subgroups are normal subgroups and maximal subgroups of arbitrary groups; moreover, Sylow subgroups of finite groups and Hall subgroups of finite soluble groups are always pronormal. The concept of a pronormal subgroup was introduced by Philip Hall, and the first results about pronormal subgroups appeared in a paper by Rose [**20**]. More recently, several researches have shown that pronormality plays a relevant role in many problems of group theory, both in the finite and the infinite case (see for instance [**9**], [**15**], [**23**]).

Received: July 27, 2017; accepted: November 7, 2017.

The authors are members of GNSAGA (INdAM), and work within the ADV-AGTA project

Pronormal subgroups are naturally related to the so-called $T$-property for a group. A group $G$ is said to have the *$T$-property* (or to be a *$T$-group*) if normality in $G$ is a transitive relation, or equivalently if all subnormal subgroups of $G$ are normal. The class of $T$-groups is not closed with respect to forming subgroups, and a group is called a *$\overline{T}$-group* if all of its subgroups have the $T$-property. Since it can be easily proved that a subgroup of a group is normal if and only if it is ascendant and pronormal, it follows that groups with only pronormal subgroups have the $\overline{T}$-property and one can expect that the behaviour of groups with many pronormal subgroups should be close to that of $\overline{T}$-groups.

The structure of groups with only pronormal subgroups was described by Kuzennyi and Subbotin [**12**]. One year later, the same authors considered in [**13**] groups all of whose infinite subgroups are pronormal, starting in this way the study of groups for which the set of non-pronormal subgroups is small in some respect. It is well-known that chain conditions often play a relevant role in the investigations concerning infinite groups, and so the next step was carried out by looking at groups satisfying either the minimal or the maximal condition on non-pronormal subgroups (see [**8**] and [**22**], respectively).

On the other hand, the imposition of weaker forms of the classical chain conditions also produces remarkable effects. In particular, Shores [**21**] and Zaicev [**24**] independently proved that if $G$ is a generalized soluble group admitting no chains of subgroups with the same order type as the set of integers, then $G$ is soluble-by-finite and it satisfies either the minimal or the maximal condition on subgroups. Conditions of this type can of course be considered for special subgroup systems. If $\theta$ is a subgroup property, we shall say that a group $G$ satisfies the *double chain condition* on $\theta$-subgroups if for each *double chain*

$$\ldots \leq X_{-n} \leq \ldots \leq X_{-1} \leq X_0 \leq X_1 \leq \ldots \leq X_n \leq \ldots$$

of $\theta$-subgroups of $G$ there exists an integer $k$ such that either $X_n = X_k$ for all $n \leq k$ or $X_n = X_k$ for all $n \geq k$. Obviously, both the minimal and the maximal conditions on $\theta$-subgroups imply the double chain condition on $\theta$-subgroups. The structure of groups satisfying the double chain conditions on normal or subnormal subgroups was studied in [**6**] and [**1**], respectively, and it should be mentioned that the double chain condition for other algebraic structures, like for instance rings and modules, has also been considered (see for instance [**3**]).

The aim of this paper is to provide a further contribution to the investigation of groups with few non-pronormal subgroups, by studying the class $DC_{np}$ of all groups satisfying the double chain condition on subgroups which are not pronormal. Among other results, it will be proved that finitely generated soluble $DC_{np}$-groups are polycyclic, and that any periodic locally soluble $DC_{np}$-group either satisfies the minimal condition on subgroups or has only pronormal subgroups. Moreover, it will be shown that a soluble $DC_{np}$-group with a torsion-free Fitting subgroup must be abelian.

Most of our notation is standard and can be found in [**18**].

## 2 - Preliminaries

Groups with the $T$-property form an important and widely investigated group class. Although all simple groups obviously have the $T$-property, the structure of soluble $T$-groups has strong restrictions and was carefully described by Gaschütz [**7**] and Robinson [**17**]. It turns out that any soluble $T$-group is metabelian and that a finitely generated soluble group with the $T$-property is either finite or abelian, so that in particular all non-periodic locally (soluble-by-finite) $\overline{T}$-groups are abelian. Notice also that finite soluble $T$-groups have the $\overline{T}$-property, and conversely any finite $\overline{T}$-group is soluble.

The first non-trivial evidence of the relation between the $T$-property and pronormality was exhibited by Peng [**16**], who proved that a finite group has the $\overline{T}$-property if and only if all of its primary subgroups are pronormal. Our first elementary results give further information on this connection.

Lemma 2.1. *Let $G$ be a group whose cyclic subgroups are pronormal. Then $G$ is a $\overline{T}$-group.*

Proof. If $g$ is any element of $G$, the cyclic subgroup $\langle g\rangle$ is pronormal in $G$ and so $G = N_G(\langle g\rangle)\langle g\rangle^G$. Then

$$\langle g\rangle^G = \langle g\rangle^{\langle g\rangle^G}$$

and hence $G$ is a $T$-group (see [**17**], Lemma 2.1.1). As the hypotheses are obviously inherited by subgroups, the group $G$ has the $\overline{T}$-property. □

Lemma 2.2. *Let $G$ be a locally finite $\overline{T}$-group. Then all finite subgroups of $G$ are pronormal.*

Proof. Let $X$ be any finite subgroup of $G$, and consider an arbitrary element $g$ of $G$. Then $\langle X, g\rangle$ is a finite $\overline{T}$-group, and so $X$ is pronormal in $\langle X, g\rangle$ (see for instance [**8**] and [**16**]). It follows that $X$ and $X^g$ are conjugate in $\langle X, X^g\rangle$, and hence $X$ is pronormal in $G$. □

Corollary 2.1. *Let $G$ be a locally (soluble-by-finite) group. Then $G$ has the $\overline{T}$-property if and only if all of its cyclic subgroups are pronormal.*

Proof. Suppose that $G$ is a $\overline{T}$-group. Then $G$ is abelian, whenever it is not periodic. On the other hand, if $G$ is periodic, then it is locally finite and so all of its cyclic subgroups are pronormal by Lemma 2.2. The converse statement holds by Lemma 2.1. □

An automorphism $\alpha$ of a group $G$ is said to be a *power automorphism* if $X^\alpha = X$ for all subgroups $X$ of $G$. The set of all power automorphisms of $G$ is an abelian and residually finite subgroup of the full automorphism group of $G$ (recall here that a group is *residually finite* if the intersection of all its subgroups of finite index is trivial). Power automorphisms have been extensively studied by Cooper [**4**]. The investigation of soluble groups with the $T$-property involves naturally and heavily the behaviour of power automorphisms. In fact, if $G$ is a $T$-group and $A$ is any abelian normal subgroup of $G$, then each element of $G$ induces by conjugation a power automorphism on $A$, so that in this situation the factor group $G/C_G(A)$ is isomorphic to a group of power automorphisms of $A$, and in particular it is abelian and residually finite.

It is useful to point out that any infinite direct decomposition of a group gives rise to a suitable double chain of subgroups which is unbounded on both sides. In fact, let $G$ be a group and let

(*) $$G = \operatorname*{Dr}_{n\in\mathbb{N}} G_n$$

be a decomposition of $G$ into the direct product of a countably infinite collection of non-trivial subgroups

$$G_1, G_2, \dots, G_n, \dots$$

Then $G$ admits the double chain

(**) $$\dots < U_{-k} < \dots < U_{-1} < U_0 < U_1 < \dots < U_k < \dots$$

where

$$U_k = \Big(\operatorname*{Dr}_{n\in\mathbb{N}} G_{2n-1}\Big) \times \Big(\operatorname*{Dr}_{1\leq n\leq k} G_{2n}\Big) \quad \text{and} \quad U_{-k} = \operatorname*{Dr}_{n>k} G_{2n-1}$$

for each non-negative integer $k$. We shall say that (**) is the *double chain associated* to the direct decomposition (*).

For basic results on pronormality, we refer to the survey paper [**10**], and we state here only the following two necessary lemmas.

Lemma 2.3. *A subgroup $X$ of a group $G$ is normal if and only if it is pronormal and ascendant. Moreover, all pronormal subgroups of a locally nilpotent group are normal.*

Lemma 2.4. *Let $G$ be a group, and let $X$ and $Y$ be pronormal subgroups of $G$ such that $X^Y = X$. Then the product $XY$ is a pronormal subgroup of $G$.*

## 3 - Main results

The first lemma of this section plays a crucial role in our arguments. Note that an *ascendant section* of an arbitrary group $G$ is a factor group of the form $X/Y$, where $X$ is an ascendant subgroup of $G$ and $Y$ is a normal subgroup of $X$.

Lemma 3.1. *Let $G$ be a $DC_{np}$-group, and let $X/Y$ be an ascendant section of $G$ which is a direct product of infinitely many cyclic non-trivial subgroups. Then:*

(a) *$X$ and $Y$ are normal in $G$.*

(b) *All cyclic subgroups of $G/X$ are pronormal.*

(c) *The factor group $G/Y$ has the $\overline{T}$-property.*

Proof. Clearly, $X/Y$ contains a normal subgroup $X^*/Y$ which is the direct product of a countable collection $(X_n/Y)_{n\in\mathbb{N}}$ of cyclic non-trivial subgroups. As $X/X^*$ is generated by cyclic normal subgroups, it is enough to prove that the statement holds for the ascendant section $X^*/Y$, and hence it can be assumed without loss of generality that $X = X^*$. Then

$$X/Y = U/Y \times V/Y,$$

where

$$U/Y = \underset{n\in\mathbb{N}}{\mathrm{Dr}}(X_{2n}/Y) \quad \text{and} \quad V/Y = \underset{n\in\mathbb{N}}{\mathrm{Dr}}(X_{2n-1}/Y).$$

As $G$ satisfies the double chain condition on non-pronormal subgroups, it follows that there exist pronormal subgroups $P$ and $Q$ of $G$ such that $Y \leq P \leq U$ and $Y \leq Q \leq V$. Since $X/Y$ is abelian, the subgroups $P$ and $Q$ are ascendant, and so even normal, in $G$. Thus $Y = P\cap Q$ is a normal subgroup of $G$. The same argument applied to the ascendant sections $X/U$ and $X/V$ shows that $U$ and $V$ are normal subgroups of $G$, so that also $X = UV$ is normal in $G$. Similarly, we obtain that

$$\langle X_i \mid i \in I\rangle$$

is a normal subgroup of $G$ for any infinite set $I$ of natural numbers, and it follows from this remark that each $X_n$ is normal in $G$.

In order to prove parts (b) and (c) of the statement, it can now obviously be assumed that $Y = \{1\}$, so that

$$X = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, X_n.$$

If $g$ is any element of $G$, there exists a positive integer $m$ such that $\langle g \rangle \cap W = \{1\}$, where

$$W = \langle X_n \mid n \geq m \rangle = \operatorname*{Dr}_{n \geq m} X_n.$$

The double chain

$$\ldots < W_{-k} < \ldots < W_{-1} < W_0 < W_1 < \ldots < W_k < \ldots$$

associated to this direct decomposition consists of normal subgroups of $G$, and so there is an integer $r$ such that the subgroup $\langle g \rangle W_r$ is pronormal in $G$. As $W_r$ is contained in $X$, it follows that also $\langle g \rangle X$ is pronormal in $G$, and hence every cyclic subgroup of $G/X$ is pronormal.

Consider now an arbitrary subgroup $K$ of $G$, and let $L$ be any subnormal subgroup of $K$. Suppose first that the subgroup $L^* = L \cap X$ is not finitely generated, so that it is the direct product of infinitely many cyclic non-trivial subgroups (see for instance [**19**], 4.3.16). In this case, it follows from parts (a) and (b) that $L^*$ is normal in $G$ and all cyclic subgroups of $G/L^*$ are pronormal; in particular, $G/L^*$ is a $\overline{T}$-group and hence $L$ is normal in $K$. Assume finally that $L^*$ is finitely generated, so that it is contained in the direct product

$$X_1 \times \ldots \times X_s$$

for some positive integer $s$. The consideration of the direct decomposition

$$\langle X_n \mid n > 2s \rangle = \big(\operatorname*{Dr}_{n>s} X_{2n}\big) \times \big(\operatorname*{Dr}_{n>s} X_{2n-1}\big)$$

and the $DC_{np}$-property yield that there exist normal subgroups $R$ and $S$ of $G$ such that $LR$ and $LS$ are pronormal in $G$ and $L \cap RS = \{1\}$. In particular, $LR$ and $LS$ are pronormal subgroups of $KR$ and $KS$, respectively. It follows that $K$ normalizes both $LR$ and $LS$, and so also $LR \cap LS = L$. Therefore $K$ is a $T$-group, and $G$ has the $\overline{T}$-property. □

Lemma 3.2. *Let $G$ be a $DC_{np}$-group, and let $X$ be a locally nilpotent subgroup of $G$. Then $X$ is either nilpotent or a Černikov group.*

Proof. It follows from Lemma 2.3 that the locally nilpotent subgroup $X$ satisfies the double chain condition on non-normal subgroups, so that $X$ satisfies either the minimal or the maximal condition on non-normal subgroups (see [**6**]). If $X$ has the minimal condition on non-normal subgroups, it is known that either $X$ is a Černikov group or all of its subgroups are normal (see [**2**]). On the other hand, when $X$ satisfies the maximal condition on non-normal subgroups, it is known that $X$ is nilpotent (see [**5**], Corollary 2.5), and so the statement is proved. □

It is well-known that in any group $G$ there exists a largest locally nilpotent normal subgroup $H(G)$, the so-called *Hirsch-Plotkin radical* of $G$, and that $H(G)$ contains all locally nilpotent ascendant subgroups of $G$. Recall also that a group is *hypercentral* if it coincides with the last term of its upper central series. Thus hypercentral groups are locally nilpotent and have only ascendant subgroups, while any locally nilpotent group satisfying the minimal condition on subgroups is hypercentral.

Corollary 3.1. *Let $G$ be a $DC_{np}$-group. Then the Hirsch-Plotkin radical $H$ of $G$ is hypercentral, and so all subgroups of $H$ are ascendant in $G$.*

Our next elementary lemma shows in particular that any finitely generated infinite abelian subgroup of a $DC_{np}$-group is contained in a finitely generated pronormal subgroup.

Lemma 3.3. *Let $G$ be a $DC_{np}$-group, and let $A$ be a finitely generated infinite abelian subgroup of $G$. Then either $A$ is pronormal in $G$ or the set of all non-pronormal subgroups of $G$ containing $A$ satisfies the maximal condition.*

Proof. Suppose that the subgroup $A$ is not pronormal in $G$, so that by Lemma 2.4 there exists an infinite sequence of positive integers

$$k_1, k_2, \dots, k_n, \dots$$

such that

$$A^{k_1} > A^{k_2} > \dots > A^{k_n} > \dots$$

and each subgroup $A^{k_n}$ is not pronormal in $G$. Since $G$ is a $DC_{np}$-group, it follows that the set of all non-pronormal subgroups of $G$ containing $A$ satisfies the maximal condition. □

We can now prove our first main result, concerning the behaviour of torsion-free subgroups in groups with the $DC_{np}$-property.

Theorem 3.1. *Let $G$ be a $DC_{np}$-group, and let $X$ be a torsion-free locally nilpotent ascendant subgroup of $G$. If $X$ is not finitely generated, then it is contained in the centre of $G$.*

Proof. Suppose first that $X$ is abelian, and assume for a contradiction that $X$ contains a finitely generated subgroup $E$ which is not normal in $G$. Then $E$ is not pronormal in $G$, so that it follows from Lemma 3.3 that the set of all subgroups of $X$ containing $E$ which are not pronormal (or equivalently not normal) in $G$ satisfies the maximal condition, and hence we may consider a finitely generated subgroup $M$ of $X$ which contains $E$ and is maximal with respect to the condition of being non-normal in $G$. Thus all subgroups of $X$ properly containing $M$ are normal in $G$, and so the intersection of any collection of non-trivial subgroups of the infinite abelian group $X/M$ is non-trivial. Therefore $X/M$ is a group of type $p^\infty$ for some prime number $p$, and $X$ has finite rank. Write

$$M = \langle a_1 \rangle \times \ldots \times \langle a_r \rangle,$$

so that both $M$ and $X$ have rank $r$. Without loss of generality, it can be assumed that the cyclic subgroup $\langle a_1 \rangle$ is not normal in $G$. Consider any prime number $q \neq p$, and put

$$M_q = \langle a_1 \rangle \times \langle a_2^q \rangle \times \ldots \times \langle a_r^q \rangle.$$

Then

$$X/M_q = M/M_q \times P_q/M_q,$$

where $P_q/M_q$ is a group of type $p^\infty$. As the infinite cyclic subgroup $\langle a_1 \rangle$ is not normal in $G$, the set of all non-normal subgroups of $X$ containing $a_1$ satisfies the maximal condition by Lemma 3.3, and hence it follows that each $P_q$ is normal in $G$. Therefore

$$P = \bigcap_{q \neq p} P_q$$

is a normal subgroup of $G$. On the other hand,

$$P \cap M = \bigcap_{q \neq p} (P_q \cap M) = \bigcap_{q \neq p} M_q = \langle a_1 \rangle,$$

so that $P$ has rank 1. Another application of Lemma 3.3 yields now that there exists a finitely generated pronormal subgroup $C$ of $G$ such that $\langle a_1 \rangle \leq C \leq P$. Then $C$ is a cyclic normal subgroup of $G$, and so also $\langle a_1 \rangle$ is normal in $G$. This contradiction shows that all subgroups of $X$ are normal in $G$, at least when $X$ is abelian.

Assume now for a contradiction that $X$ is not contained in $Z(G)$, again under the assumption that $X$ is abelian, and let $g$ be an element of $G$ such that $[X, g] \neq \{1\}$. Since all subgroups of $X$ are normal in $G$, we have that $g$ induces by conjugation the inversion automorphism on $X$, so that $[X, g^2] = \{1\}$ and $X \cap \langle g \rangle = \{1\}$. If the subgroup $X^4$ is not finitely generated, there exist infinitely many elements

$$b_1, b_2, \ldots, b_n, \ldots$$

of $X$ such that

$$\ldots < \langle g, b_1^{4n} \rangle < \ldots < \langle g, b_1^4 \rangle < \langle g, b_1^4, b_2^4 \rangle < \ldots < \langle g, b_1^4, b_2^4, \ldots, b_n^4 \rangle < \ldots$$

and it follows from the $DC_{np}$-property that $X$ contains a finitely generated subgroup $U$ such that $\langle g, U^4 \rangle$ is pronormal in $G$. On the other hand, the factor group $\langle g, U \rangle / \langle g^2, U^4 \rangle$ is nilpotent (see [**18**] Part 2, Lemma 6.34), so that $\langle g, U^4 \rangle$ is normal in $\langle g, U \rangle$ and hence

$$\langle g, U \rangle / \langle g^2, U^4 \rangle = \langle g, U^4 \rangle / \langle g^2, U^4 \rangle \times \langle g^2, U \rangle / \langle g^2, U^4 \rangle,$$

which is impossible because $U/U^4$ has exponent 4. Thus $X^4$ must be finitely generated, so that $X/X^4$ is infinite of exponent 4 and hence

$$X/X^4 = V/X^4 \times W/X^4,$$

where $V/X^4$ is cyclic of order 4 and $W/X^4$ can be decomposed into the direct product of infinitely many cyclic subgroups. The consideration of the double chain associated to this decomposition allows to consider a subgroup $Y$ such that $X^4 \leq Y \leq W$ and $\langle g, Y \rangle$ is pronormal in $G$. It follows that $\langle g, W \rangle$ is pronormal in $G$, so that $\langle g, W \rangle / \langle g^2, W \rangle$ is a normal subgroup of the finite 2-group $\langle g, X \rangle / \langle g^2, W \rangle$ and hence

$$\langle g, X \rangle / \langle g^2, W \rangle = \langle g, W \rangle / \langle g^2, W \rangle \times \langle g^2, X \rangle / \langle g^2, W \rangle,$$

which is impossible, as $X/W$ is cyclic of order 4. This further contradiction proves that $X \leq Z(G)$, provided that $X$ is abelian.

Suppose finally that $X$ is locally nilpotent, so that it is even nilpotent by Lemma 3.2. Since $X$ is not finitely generated, it cannot satisfy the maximal condition on abelian subgroups and so it contains a maximal abelian subgroup $A$ which is not finitely generated. Clearly, $A$ is subnormal in $G$ and so it is contained in $Z(G)$ by the first part of the proof. Thus

$$X = C_X(A) = A,$$

and the proof is complete. □

Recall that a group $G$ is said to be *radical* if it has an ascending series with locally nilpotent factors. Thus all soluble groups are radical, and a finite group is radical if and only if it is soluble. Moreover, in any radical group the Hirsch-Plotkin radical contains its centralizer.

Corollary 3.2. *Let $G$ be a radical $DC_{np}$-group whose Hirsch-Plotkin radical is torsion-free. Then $G$ is either polycyclic or abelian.*

Proof. Assume that $G$ is not polycyclic. Then the Hirsch-Plotkin radical $H$ of $G$ cannot be finitely generated (see [**18**] Part 1, Theorem 3.27), and so it follows from Theorem 3.1 that $H$ is contained in $Z(G)$. Therefore

$$G = C_G(H) = H$$

is an abelian group. □

Our next result is another consequence of Theorem 3.1, and shows that within the universe of $DC_{np}$-groups the property of being radical is equivalent to solubility. It proves also that in the statement of Corollary 3.2 the Hirsch-Plotkin radical may be replaced by the Fitting subgroup.

Corollary 3.3. *Let $G$ be a radical $DC_{np}$-group. Then $G$ is soluble.*

Proof. Let $T$ be the largest periodic normal subgroup of $G$. Then the factor group $G/T$ has a torsion-free Hirsch-Plotkin radical, so that $G/T$ is either polycyclic or abelian by Corollary 3.2, and hence it is soluble. Thus it is enough to show that $T$ is soluble, which is of course the case when $T$ is a Černikov group. Suppose now that $T$ is not a Černikov group, so that its Hirsch-Plotkin radical $H$ does not satisfy the minimal condition on abelian subgroups (see [**18**] Part 1, Theorem 3.32), and so it contains a subgroup $A$ which is the direct product of infinitely many cyclic non-trivial subgroups. Moreover, as $H$ is nilpotent by Lemma 3.2, the subgroup $A$ is subnormal in $G$, so that it follows from Lemma 3.1 that $G$ is a $\overline{T}$-group and hence it is metabelian. The statement is proved. □

Our next statement should be related to the already mentioned fact that any finitely generated soluble $T$-group is either finite or abelian.

Theorem 3.2. *Let $G$ be a finitely generated soluble $DC_{np}$-group. Then $G$ is polycyclic.*

Proof. Let $T$ be the largest periodic normal subgroup of $G$. Then $G/T$ has a torsion-free Hirsch-Plotkin radical, and so it follows from Corollary 3.2 that the factor group $G/T$ is polycyclic. Assume for a contradiction that $G$ is not polycyclic, so that the subgroup $T$ is infinite. If the Hirsch-Plotkin radical $H$ of $T$ satisfies the minimal condition on abelian subgroups, we have that $T$ is a Černikov group; in this case, $G$ is abelian-by-polycyclic, so that it is residually finite (see [**11**]) and hence $T$ is finite, a contradiction. Therefore $H$ contains a subgroup $A$ which is the direct product of an infinite collection of cyclic non-trivial subgroups. As $A$ is ascendant in $G$, it follows from Lemma 3.1 that the group $G$ has the $\overline{T}$-property, so that it is either finite or abelian, the final contradiction. □

We turn now to the case of periodic $DC_{np}$-groups. The following lemma proved by Zaicev [**25**] is relevant for our purposes.

L e m m a 3.4. *Let $G$ be a periodic locally soluble group. If there exists a finite group of automorphisms $\Gamma$ of $G$ such that all abelian $\Gamma$-invariant subgroups of $G$ satisfy the minimal condition on subgroups, then $G$ satisfies the minimal condition on abelian subgroups.*

T h e o r e m 3.3. *Let $G$ be a periodic locally soluble $DC_{np}$-group. Then either $G$ is a Černikov group or all of its subgroups are pronormal.*

P r o o f. Suppose that $G$ is not a Černikov group, so that it cannot satisfy the minimal condition on abelian subgroups (see [**18**] Part 1, Theorem 3.45). If $E$ is an arbitrary finite subgroup of $G$, it follows from Lemma 3.4 that $G$ contains an abelian subgroup $A$ admitting a direct decomposition

$$A = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, A_n,$$

where each $A_n$ is a finite non-trivial $E$-invariant subgroup, and of course $A$ can be chosen in such a way that $A \cap E = \{1\}$. Let $X$ be any subnormal subgroup of $E$. An application of the $DC_{np}$-property yields that there exists an $E$-invariant subgroup $B$ of $A$ such that the product $XB$ is a pronormal subgroup of $G$. In particular, the subgroup $XB$ is pronormal and subnormal in $EB$, so that it is normal in $EB$. Thus $X = XB \cap E$ is normal in $E$, and hence $E$ is a $T$-group. It follows that the locally finite group $G$ has the $\overline{T}$-property, so that in particular $G$ is metabelian and by Lemma 2.2 all of its finite subgroups are pronormal.

Assume now for a contradiction that $G$ contains subgroups which are not pronormal, so that in particular it cannot satisfy the maximal condition on non-pronormal subgroups (see [**22**]) and hence we may consider a minimal non-pronormal subgroup $M$ of $G$. Then an application of Lemma 2.4 yields that $M$ cannot be decomposed into a product $UV$, where $U$ and $V$ are proper subgroups and $U^V = U$. As $M$ is obviously infinite, it follows that $M/M'$ is a group of type $p^\infty$ for some prime number $p$. Moreover, since $M$ has the $T$-property, the factor group $M/C_M(M')$ is residually finite, so that $M'$ is contained in $Z(M)$. Thus $M' = \{1\}$ and $M$ is a group of type $p^\infty$. Let $H$ be the Hirsch-Plotkin radical of $G$. Then $H$ is nilpotent by Lemma 3.2 and $G/C_G(H)$ is residually finite, because $G$ is a $T$-group, so that

$$M \leq C_G(H) \leq H$$

and $M$ is normal in $G$. This contradiction completes the proof. □

Recall finally that a group $G$ is *minimax* if it has a series of finite length each of whose factors satisfies either the minimal or the maximal condition on subgroups. The structure of soluble minimax groups has been described by Robinson (see [**18**] Part 2, Chapter 10).

Lemma 3.5. *Let $G$ be a locally soluble $DC_{np}$-group whose Hirsch-Plotkin radical is not minimax. Then $G$ has the $\overline{T}$-property.*

Proof. Assume for a contradiction that $G$ is not a $\overline{T}$-group, and let $H$ be the Hirsch-Plotkin radical of $G$. Then $H$ is nilpotent by Lemma 3.2, so that it has the double chain condition on non-normal subgroups. It follows that $H$ satisfies either the minimal or the maximal condition on non-normal subgroups (see [**6**]), and it is known that in both cases $H$ is a Dedekind group (see [**2**] and [**5**]). On the other hand, an application of Lemma 3.1 yields that the largest periodic subgroup $T$ of $H$ satisfies the minimal condition on subgroups, so that $H$ is a non-periodic abelian group and $H = T \times K$, where $K$ is a torsion-free subgroup which cannot be minimax. Let $L$ be a free abelian subgroup of $K$ such that $K/L$ is periodic. A further application of Lemma 3.1 shows that $L$ is finitely generated, so that $K/L$ does not satisfy the minimal condition on subgroups. Then for each positive integer $n$ the group $K/L^n$ contains a subgroup which is the direct product of infinitely many cyclic non-trivial subgroups, and hence it follows again from Lemma 3.1 that $G/L^n$ is a $\overline{T}$-group. In particular, $G/L^n$ is metabelian for all $n$, so that

$$G'' \leq \bigcap_{n\in\mathbb{N}} L^n = \{1\}.$$

Then $G$ is metabelian, and hence $C_G(H) \leq H$. On the other hand, $K$ is contained in $Z(G)$ by Theorem 3.1, so that $G$ cannot act by conjugation on $H$ as a group of power automorphisms and $H$ contains an infinite cyclic subgroup $\langle h\rangle$ which is not normal in $G$. As $H$ is not finitely generated, by Lemma 3.3 it contains a finitely generated subgroup $M$ which is maximal with respect to the condition of being non-normal in $G$. It follows that the identity subgroup cannot be realized as intersection of non-trivial subgroups of the abelian group $H/M$, so that $H/M$ satisfies the minimal condition on subgroups and hence $H$ is minimax. This last contradiction completes the proof of the statement. □

Theorem 3.4. *Let $G$ be a soluble $DC_{np}$-group. Then either $G$ is minimax or it has the $\overline{T}$-property.*

Proof. Suppose that $G$ is not a $\overline{T}$-group. Then the Hirsch-Plotkin radical of $G$ is minimax by Lemma 3.5. In particular, all abelian ascendant subgroups of $G$ are minimax, and hence $G$ itself is a minimax group by a result of Baer (see [**14**], Corollary 6.3.9). □

## References

[**1**] M. Brescia and F. de Giovanni, *Groups satisfying the double chain condition on subnormal subgroups*, Ric. Mat. **65** (2016), 255–261.

[**2**] S. N. Černikov, *Infinite nonabelian groups with a minimal condition for non-normal subgroups*, Math. Notes **6** (1969), 465–468.

[**3**] M. Contessa, *On rings and modules with DICC*, J. Algebra **101** (1986), 489–496.

[**4**] C. D. H. Cooper, *Power automorphisms of groups*, Math. Z. **107** (1968), 335–356.

[**5**] G. Cutolo, *On groups satisfying the maximal condition on nonnormal subgroups*, Riv. Mat. Pura Appl. **9** (1991), 49–59.

[**6**] F. De Mari and F. de Giovanni, *Double chain conditions for infinite groups*, Ricerche Mat. **54** (2005), 59–70.

[**7**] W. Gaschütz, *Gruppen, in denen das Normalteilersein transitiv ist.*, J. Reine Angew. Math. **198** (1957), 87–92.

[**8**] F. de Giovanni and G. Vincenzi, *Groups satisfying the minimal condition on non-pronormal subgroups*, Boll. Un. Mat. Ital. A (7) **9** (1995), 185–194.

[**9**] F. de Giovanni and G. Vincenzi, *Pronormality in infinite groups*, Math. Proc. R. Ir. Acad. **100A** (2000), 189–203.

[**10**] F. de Giovanni and G. Vincenzi, *Some topics in the theory of pronormal subgroups of groups*, in: "Topics in infinite groups", Quad. Mat., **8**, Seconda Univ. Napoli, Caserta, 2001, 175–202.

[**11**] A. V. Jategaonkar, *Integral group rings of polycyclic-by-finite groups*, J. Pure Appl. Algebra **4** (1974), 337–343.

[**12**] N. F. Kuzennyi and I. Ya. Subbotin, *Groups in which all subgroups are pronormal*, Ukrain. Math. J. **39** (1987), 251–254.

[**13**] N. F. Kuzennyi and I. Ya. Subbotin, *Locally soluble groups in which all infinite subgroups are pronormal*, Soviet Math. (Iz. VUZ) **32** (1988), 126–131.

[**14**] J. C. Lennox and D. J. S. Robinson, *The theory of infinite soluble groups*, The Clarendon Press, Oxford, 2004.

[**15**] G. Navarro, *Pronormal subgroups and zeros of characters*, Proc. Amer. Math. Soc. **142** (2014), 3003–3005.

[**16**] T. A. Peng, *Finite groups with pro-normal subgroups*, Proc. Amer. Math. Soc. **20** (1969), 232–234.

[**17**] D. J. S. Robinson, *Groups in which normality is a transitive relation*, Proc. Cambridge Philos. Soc. **60** (1964), 21–38.

[**18**] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups, Part* 1 & *Part* 2, Springer-Verlag, Berlin 1972.

[**19**] D. J. S. Robinson, *A course in the theory of groups*, 2nd edition, Grad. Texts in Math., **80**, Springer-Verlag, New York, 1996.

[**20**] J. S. Rose, *Finite soluble groups with pronormal system normalizers*, Proc. London Math. Soc. (3) **17** (1967), 447–469.

[**21**] T. S. Shores, *A chain condition for groups*, Rocky Mountain J. Math. **3** (1973), 83–89.

[**22**] G. Vincenzi, *Groups satisfying the maximal condition on non-pronormal subgroups*, Algebra Colloq. **5** (1998), 121–134.

[**23**] G. J. Wood, *On pronormal subgroups of finite soluble groups*, Arch. Math. (Basel) **25** (1974), 578–585.

[**24**] D. I. Zaicev, *Theory of minimax groups*, Ukrainian Math. J. **23** (1971), 536–542.

[**25**] D. I. Zaicev, *On solvable subgroups of locally solvable groups*, Soviet Math. Dokl. **15** (1974), 342–345.

Mattia Brescia
Università di Napoli Federico II
Dipartimento di Matematica e Applicazioni
via Cintia
Napoli, 80126, Italy
e-mail: mattia.brescia@unina.it

Francesco de Giovanni
Università di Napoli Federico II
Dipartimento di Matematica e Applicazioni
via Cintia
Napoli, 80126, Italy
e-mail: degiovan@unina.it