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Timelike surfaces with a common asymptotic curve in Minkowski 3-space

Abstract. In this paper, we study the problem of constructing a timelike surface pencil from a given spacelike or timelike asymptotic curve in Minkowski 3-space $\mathbb{E}_1^3$. Using the Serret–Frenet frame of the given spacelike or timelike asymptotic curve, we present the timelike surface as a linear combination of this frame and analyze the necessary and sufficient condition for that curve to be asymptotic. We illustrate this method by presenting some examples.

Keywords. Serret–Frenet formulae, Marching-scale functions, Spacelike and timelike asymptotic curves.

Mathematics Subject Classification (2010):53B30, 51B20, 53C50.

1 - Introduction

In differential geometry of surfaces, an asymptotic curve is a curve always tangent to an asymptotic direction of the surface (where they exist). It is sometimes called an asymptotic line, although it need not be a line. An asymptotic direction is one in which the normal curvature is zero. Which is to say: for a point on an asymptotic curve, take the plane which bears both the curve's tangent and the surface's normal at that point. The curve of intersection of the plane and the surface will have zero curvature at that point. Asymptotic directions can only occur when the Gaussian curvature is negative (or zero). There will be two asymptotic directions through every point with negative Gaussian curvature, these directions are bisected by the principal directions [**2**, **3**]. In Euclidean 3-space $\mathbb{E}^3$, surfaces with common asymptotic curve have been the

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subject of many studies. For example, Pottmann et al. [**9**] proposed a geometry-processing framework to approximate a given shape by one or more strips of ruled surfaces. Flory and Pottmann [**4**] addressed challenges in the realization of free-form architecture and complex shapes in general with the technical advantages of ruled surfaces. In that work, they used asymptotic curves obtained by careful investigation and constructed an initial ruled surface by aligning the rulings with asymptotic curves; they also discussed how the shape of this initial approximation can be modified to optimally fit a given target shape. In practical applications, the concept of family of surfaces having a given characteristic curve was first introduced by Wang et.al. [**13**] in Euclidean 3-space. The basic idea is to regard the wanted surface as an extension from the given characteristic curve, and represent it as a linear combination of the marching-scale functions $u(s,t)$, $v(s,t)$, $w(s,t)$ and the three vector functions $\mathbf{t}(s)$, $\mathbf{n}(s)$, $\mathbf{b}(s)$, which are the unit tangent, the principal normal and the binormal vector of the curve respectively. With the given characteristic curve and isoparametric constraints, they derived the necessary and sufficient conditions for the correct parametric representation of the surface pencil. This principal has been used treated extensively in the works [**1**, **6**, **7**]. In a somewhat parallel fashion, Minkowski space is the basis for the study of the physical phenomena described by the theory of relativity which has great geometric and physical meaning. Much work to date, therefore, has been done on timelike and spacelike surfaces in $\mathbb{E}_1^3$ (the three-dimensional Minkowski space). In [**5**] Kasap and Akyıldız defined surfaces with a common geodesic in Minkowski 3-space and gave the sufficient conditions on marching-scale functions so that the given curve is a common geodesic on that surfaces. Saffak and Kasap [**11**] studied family of surfaces with a common null geodesic. Recently, in [**10**] Saffak et.al studied family of surfaces with a common spacelike (timelike) asymptotic curve using the Serret–Frenet frame of the curve.

In this paper, we extend the method of Wang et al. [**13**] to derive the necessary and sufficient condition for a given spacelike or timelike curve to be both isoparametric and asymptotic on a timelike surface. Thus, we define family of timelike surfaces with a common asymptotic curve. Also we show with helps of given examples that the member, having any desired property, can be obtained by choosing the appropriate coefficients.

2 - Preliminaries

Let $\mathbb{E}_1^3$ be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{R}^3$ with the metric

$$< d\mathbf{x}, d\mathbf{x} >= dx_1^2 + dx_2^2 - dx_3^2$$

where (x_1, x_2, x_3) denotes the canonical coordinates in $\mathbb{R}^3$. An arbitrary vector $\mathbf{x}$ of $\mathbb{E}_1^3$ is said to be spacelike if $< \mathbf{x}, \mathbf{x} > > 0$ or $\mathbf{x} = \mathbf{0}$, timelike if $< \mathbf{x}, \mathbf{x} > < 0$ and lightlike or null if $< \mathbf{x}, \mathbf{x} >= 0$ and $\mathbf{x} \neq \mathbf{0}$. A timelike or light-like vector in $\mathbb{E}_1^3$ is said to be causal. For $\mathbf{x} \in \mathbb{E}_1^3$ the norm is defined by $\|\mathbf{x}\| = \sqrt{|< \mathbf{x}, \mathbf{x} >|}$, then the vector $\mathbf{x}$ is called a spacelike unit vector if $< \mathbf{x}, \mathbf{x} >= 1$ and a timelike unit vector if $< \mathbf{x}, \mathbf{x} >= -1$. Similarly, a regular curve in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [**12**]. For any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ of $\mathbb{E}_1^3$, the inner product is the real number $< \mathbf{x}, \mathbf{y} >= x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by $\mathbf{x} \times \mathbf{y} = ((x_2y_3 - x_3y_2), (x_3y_1 - x_1y_3), -(x_1y_2 - x_2y_1))$. Let $\alpha = \alpha(s)$ be a unit speed spacelike curve in $\mathbb{E}_1^3$; by $\kappa(s)$ and $\tau(s)$ we denote the natural curvature and torsion of $\alpha = \alpha(s)$, respectively. Consider the Serret–Frenet frame $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ associated with curve $\alpha = \alpha(s)$ such that $\mathbf{t}(s)$ and $\mathbf{b}(s)$ are spacelike vectors while $\mathbf{n}(s)$ is timelike vector, then the Serret–Frenet formulae read [**8, 12**]:

$$\frac{d}{ds}\begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}, \tag{1}$$

where

$$\mathbf{t}(s) \times \mathbf{n}(s) = \mathbf{b}(s),\ \mathbf{t}(s) \times \mathbf{b}(s) = \mathbf{n}(s),\ \mathbf{n}(s) \times \mathbf{b}(s) = \mathbf{t}(s). \tag{2}$$

If $\alpha = \alpha(s)$ is a unit speed timelike curve, then above equations are given as:

$$\frac{d}{ds}\begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}, \tag{3}$$

and

$$\mathbf{t}(s) \times \mathbf{n}(s) = \mathbf{b}(s),\ \mathbf{t}(s) \times \mathbf{b}(s) = -\mathbf{n}(s),\ \mathbf{n}(s) \times \mathbf{b}(s) = -\mathbf{t}(s). \tag{4}$$

Excluding the case in which the curvature vanishes because the second derivative of the curve $\alpha(s)$ is zero at some set of points.

Similar with [**13**], to construct a timelike surface pencil that possess $\alpha(s)$ as a common spacelike/timelike curve, we give the parametric form of the surface $\mathbf{P}(s,t)$: $[0,L]\times[0,T]\to\mathbb{E}_1^3$ as follows:

$$\mathbf{P}(s,t)=\alpha(s)+u(s,t)\mathbf{t}(s)+v(s,t)\mathbf{n}(s)+w(s,t)\mathbf{b}(s),\ 0\le t_0\le T,\ \ 0\le s\le L, \tag{5}$$

where $u(s,t)$, $v(s,t)$ and $w(s,t)$ are all C^1 functions. If the parameter t is seen as the time, the functions $u(s,t)$, $v(s,t)$ and $w(s,t)$ can then be viewed as directed marching distances of a point unit in the time t in the direction $\mathbf{t}(s)$, $\mathbf{n}(s)$ and $\mathbf{b}(s)$, respectively, and the position vector $\alpha(s)$ is seen as the initial location of this point. From now on, we shall often not write the parameters s, and t explicitly in the functions $u(s,t)$, $v(s,t)$ and $w(s,t)$. Depending on the causal character of the curve $\alpha=\alpha(s)$, the normal vector field is given by

$$\mathbf{N}(s,t):=\frac{\partial\mathbf{P}(s,t)}{\partial s}\times\frac{\partial\mathbf{P}(s,t)}{\partial t}=\eta_1(s,t)\mathbf{t}(s)+\eta_2(s,t)\mathbf{n}(s)+\eta_3(s,t)\mathbf{b}(s), \tag{6}$$

where

$$\left.\begin{aligned}\eta_1(s,t)&=\left(\frac{\partial v}{\partial s}+u\kappa(s)+w\tau(s)\right)\frac{\partial w}{\partial t}-\left(\frac{\partial w}{\partial s}+v\tau(s)\right)\frac{\partial v}{\partial t},\\ \eta_2(s,t)&=\left(1+\frac{\partial u}{\partial s}+v\kappa(s)\right)\frac{\partial w}{\partial t}-\left(\frac{\partial w}{\partial s}+v\tau(s)\right)\frac{\partial u}{\partial t},\\ \eta_3(s,t)&=\left(1+\frac{\partial u}{\partial s}+v\kappa(s)\right)\frac{\partial v}{\partial t}-\left(\frac{\partial v}{\partial s}+u\kappa(s)+w\tau(s)\right)\frac{\partial u}{\partial t},\end{aligned}\right\} \tag{7}$$

if $\alpha=\alpha(s)$ is a spacelike curve, and

$$\left.\begin{aligned}\eta_1(s,t)&=-\left(\frac{\partial v}{\partial s}+u\kappa(s)-w\tau(s)\right)\frac{\partial w}{\partial t}+\left(\frac{\partial w}{\partial s}+v\tau(s)\right)\frac{\partial v}{\partial t},\\ \eta_2(s,t)&=-\left(1+\frac{\partial u}{\partial s}+v\kappa(s)\right)\frac{\partial w}{\partial t}+\left(\frac{\partial w}{\partial s}+v\tau(s)\right)\frac{\partial u}{\partial t},\\ \eta_3(s,t)&=\left(1+\frac{\partial u}{\partial s}+v\kappa(s)\right)\frac{\partial v}{\partial t}-\left(\frac{\partial v}{\partial s}+u\kappa(s)-w\tau(s)\right)\frac{\partial u}{\partial t},\end{aligned}\right\} \tag{8}$$

if $\alpha=\alpha(s)$ is a timelike curve.

$\mathbf{P}(s,t)$ is called a spacelike/timelike surface if the induced metric is a Riemannian/Lorentzian metric on each tangent plane [**8**]. This is equivalent to saying that the normal vector $\mathbf{N}$ is timelike/spacelike at each point of $\mathbf{P}(s,t)$. A curve on a surface is asymptotic if and only if the binormal vector to the curve is everywhere parallel to the local normal vector of the surface [**2**,**3**]. An isoparametric curve is a curve $\alpha = \alpha(s)$ on a surface $\mathbf{P}(s,t)$ in $\mathbb{E}^3_1$ that has a constant s or $t-$parameter value. In other words, there exists a parameter s_0 or t_0 such that $\alpha(s) = \mathbf{P}(s,t_0)$or $\alpha(t) = \mathbf{P}(s_0,t)$. Given a parametric curve $\alpha = \alpha(s)$, we call it an isoasymptotic of the surface $\mathbf{P}(s,t)$ if it is both an asymptotic curve and a parameter curve on $\mathbf{P}(s,t)$ [**10**].

3 - Timelike Surfaces with a common asymptotic curve

Our goal is to derive necessary and sufficient conditions for which the given spacelike or timelike curve $\alpha(s)$ is an isoparametric and asymptotic (asymptotic for short) on the timelike surface $\mathbf{P}(s,t)$. First, since the directrix $\alpha(s)$ is an isoparametric curve on the surface there exists a parameter $t = t_0 \in [0,T]$ such that $\alpha(s) = \mathbf{P}(s,t_0)$, that is, we have

$$u(s,t_0) = v(s,t_0) = w(s,t_0) = 0. \tag{9}$$

Depending on the causal character of the curve $\alpha = \alpha(s)$ we obtain two timelike surfaces:

3.1 - $\alpha = \alpha(s)$ *is a spacelike asymptotic curve*

Via Eq. (9), Eq. (6) become:

$$\mathbf{N}(s,t_0) = \eta_1(s,t_0)\mathbf{t}(s) + \eta_2(s,t_0)\mathbf{n}(s) + \eta_3(s,t_0)\mathbf{b}(s), \tag{10}$$

where

$$\left.\begin{aligned} \eta_1(s,t_0) &= \frac{\partial v(s,t)}{\partial s}\frac{\partial w(s,t)}{\partial t} - \frac{\partial w(s,t)}{\partial s}\frac{\partial v(s,t)}{\partial t},\\ \eta_2(s,t_0) &= \left(1+\frac{\partial u(s,t)}{\partial s}\right)\frac{\partial w(s,t)}{\partial t} - \frac{\partial w(s,t)}{\partial s}\frac{\partial u(s,t)}{\partial t},\\ \eta_3(s,t_0) &= \left(1+\frac{\partial u(s,t)}{\partial s}\right)\frac{\partial v(s,t)}{\partial t} - \frac{\partial v(s,t)}{\partial s}\frac{\partial u(s,t)}{\partial t}. \end{aligned}\right\} \tag{11}$$

Second, the curve $\alpha(s)$ is an asymptotic curve on the surface $\mathbf{P}(s,t)$ if and only if at any point on the curve $\alpha(s)$ the binormal $\mathbf{b}(s)$ to the curve and the normal

$\mathbf{N}(s,t_0)$ to the surface $\mathbf{P}(s,t)$ are parallel to each other [**2**, **3**]. Thus, we have that $\mathbf{b}(s)\|\mathbf{N}(s,t_0)$ if and only if

$$\eta_1(s,t_0)=0,\quad \eta_2(s,t_0)=0,\quad \eta_3(s,t_0)\neq 0. \tag{12}$$

Hence, combining the conditions (9) and (12), the following theorem is proved:

Theorem 3.1. *A given spacelike curve $\alpha(s)$ is an asymptotic curve on the timelike surface $\mathbf{P}(s,t)$ if and only if*

$$\left.\begin{aligned}
u(s,t_0)=v(s,t_0)=w(s,t_0)&=0,\\
\eta_1(s,t_0)=\frac{\partial v(s,t)}{\partial s}\frac{\partial w(s,t)}{\partial t}-\frac{\partial w(s,t)}{\partial s}\frac{\partial v(s,t)}{\partial t}&=0,\\
\eta_2(s,t_0)=\left(1+\frac{\partial u(s,t)}{\partial s}\right)\frac{\partial w(s,t)}{\partial t}-\frac{\partial w(s,t)}{\partial s}\frac{\partial u(s,t)}{\partial t}&=0,\\
\eta_3(s,t_0)=\left(1+\frac{\partial u(s,t)}{\partial s}\right)\frac{\partial v(s,t)}{\partial t}-\frac{\partial v(s,t)}{\partial s}\frac{\partial u(s,t)}{\partial t}&\neq 0.
\end{aligned}\right\} \tag{13}$$

We call the set of timelike surfaces defined by Eqs. (6) and (13) the family of timelike surfaces with common spacelike asymptotic curve. Any timelike surface $\mathbf{P}(s,t)$ defined by Eq. (6) and satisfying Eq. (13) is a member of this family. In Ref. [**13**], for the purposes of simplifications and analysis, the marching-scale functions $u(s,t)$, $v(s,t)$ and $w(s,t)$ were decomposed into two factors:

$$\begin{aligned}
u(s,t)&=l(s)U(t),\\
v(s,t)&=m(s)V(t),\\
w(s,t)&=n(s)W(t).
\end{aligned} \tag{14}$$

Here $l(s),m(s),n(s),U(t),V(t)$ and $W(t)$ are C^1 functions and $l(s),m(s)$ and $n(s)$ are not identically zero. Thus, from the Theorem 3.1, we can get the following corollary:

Corollary 3.1. *The necessary and sufficient condition for the spacelike curve $\alpha(s)$ being an asymptotic curve on the timelike surface $\mathbf{P}(s,t)$ is*

$$\left.\begin{gathered}
U(t_0)=V(t_0)=W(t_0)=0,\\
\frac{dW(t_0)}{dt}=0,\ \textit{or}\ n(s)=0,\\
\frac{dV(t_0)}{dt}=\textit{const}\neq 0,\ \textit{and}\ m(s)\neq 0,\\
0\le t_0\le T,\ 0\le s\le L.
\end{gathered}\right\} \tag{15}$$

Now, let us consider other types of marching-scale functions. In Eq. (6), $u(s,t)$, $v(s,t)$ and $w(s,t)$ functions can be chosen in two different forms:
(1) If we choose

$$(16)\qquad \begin{cases} u(s,t) = \sum_{k=1}^{p} a_{1k} l(s)^k U(t)^k, \\ v(s,t) = \sum_{k=1}^{p} a_{2k} m(s)^k V(t)^k, \\ w(s,t) = \sum_{k=1}^{p} a_{3k} n(s)^k W(t)^k, \end{cases}$$

then we can simply express the sufficient condition for which the curve $\alpha(s)$ being a common asymptotic spacelike curve on the timelike surface $\mathbf{P}(s,t)$ as

$$(17)\qquad \begin{cases} U(t_0) = V(t_0) = W(t_0) = 0, \\ a_{21} \neq 0, \ \dfrac{dV(t_0)}{dt} = const \neq 0, \text{ and } m(s) \neq 0, \\ a_{31} = 0, \text{ or } \dfrac{dW(t_0)}{dt} = 0, \text{ or } n(s) = 0, \end{cases}$$

where $l(s)$, $m(s)$, $n(s)$, $U(t)$, $V(t)$ and $W(t)$ are C^1 functions, $a_{ij} \in \mathbb{R}$ $(i = 1,2,3; j = 1,2,...,p)$ and $l(s)$, $m(s)$ and $n(s)$ are not identically zero.
(2) If we choose

$$(18)\qquad \begin{cases} u(s,t) = f\Big(\sum_{k=1}^{p} a_{1k} l(s)^k U(t)^k\Big), \\ v(s,t) = g\Big(\sum_{k=1}^{p} a_{2k} m(s)^k V(t)^k\Big), \\ w(s,t) = h\Big(\sum_{k=1}^{p} a_{3k} n(s)^k W(t)^k\Big), \end{cases}$$

then we can rewrite the Eq. (13), for which the spacelike curve $\alpha = \alpha(s)$ being a common asymptotic spacelike curve on the timelike surface $\mathbf{P}(s,t)$, as

$$(19)\qquad \begin{cases} U(t_0) = V(t_0) = W(t_0) = f(0) = g(0) = h(0) = 0, \\ a_{21} \neq 0, \ \dfrac{dV(t_0)}{dt} = const \neq 0, \ m(s) \neq 0, \text{ and } g^{'}(0) \neq 0, \\ a_{31} = 0, \text{ or } \dfrac{dW(t_0)}{dt} = 0, \text{ or } n(s) = 0, \text{ or } h^{'}(0) = 0, \end{cases}$$

where $U(t)$, $V(t)$, $W(t)$, $l(s)$, $m(s)$, $n(s)$, f, g and h are C^1 functions. Since there are no constraints related to the given curve in Eqs. (15), (17) or (19), the surface pencil with the spacelike curve $\alpha(s)$ as the common asymptotic, can always be found by choosing suitable marching-scale functions. In what follows some representative examples are illustrated to verify the method.

Example 3.1. In this example, we will construct timelike surface pencil in which all the surfaces share an asymptotic spacelike helix represented as:

$$\alpha(s) = \Big(a\sinh\frac{s}{c}, \frac{bs}{c}, a\cosh\frac{s}{c}\Big), \quad -2 \le s \le 2,$$

where $a, b, c \in \mathbb{R}$, and $a^2 + b^2 = c^2$. It is easy to show that

$$\left.\begin{aligned} \mathbf{t}(s) &= \Big(\frac{a}{c}\cosh\frac{s}{c}, \frac{b}{c}, \frac{a}{c}\sinh\frac{s}{c}\Big), \\ \mathbf{n}(s) &= \Big(\sinh\frac{s}{c}, 0, \cosh\frac{s}{c}\Big), \\ \mathbf{b}(s) &= \Big(\frac{b}{c}\cosh\frac{s}{c}, -\frac{a}{c}, \frac{b}{c}\sinh\frac{s}{c}\Big). \end{aligned}\right\}$$

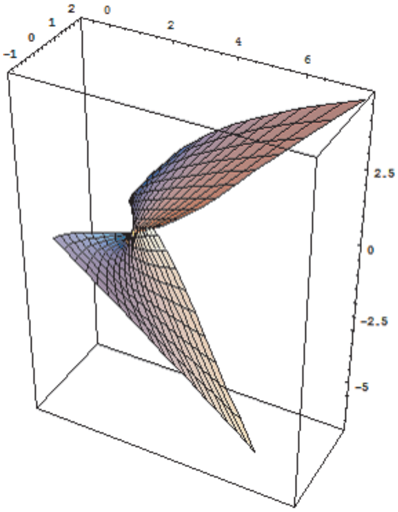
1 - By choosing $u(s,t) = 0$, $v(s,t) = \beta t$, $w(s,t) = \gamma t^2$, $t_0 = 0$, where $\beta, \gamma \in \mathbb{R}$, $\beta \neq 0$, and $0 \le t \le 2$, then Eq. (15) is satisfied. Thus, we obtain the timelike surface pencil with a common spacelike asymptotic curve $\alpha(s)$ as

$$\mathbf{P}(s,t;\beta,\gamma) = \Big(a\sinh\frac{s}{c}, \frac{bs}{c}, a\cosh\frac{s}{c}\Big) + t(0,\beta,\gamma t)\begin{pmatrix} \frac{a}{c}\cosh\frac{s}{c} & \frac{b}{c} & \frac{a}{c}\sinh\frac{s}{c} \\ \sinh\frac{s}{c} & 0 & \cosh\frac{s}{c} \\ \frac{b}{c}\cosh\frac{s}{c} & -\frac{a}{c} & \frac{b}{c}\sinh\frac{s}{c} \end{pmatrix}.$$

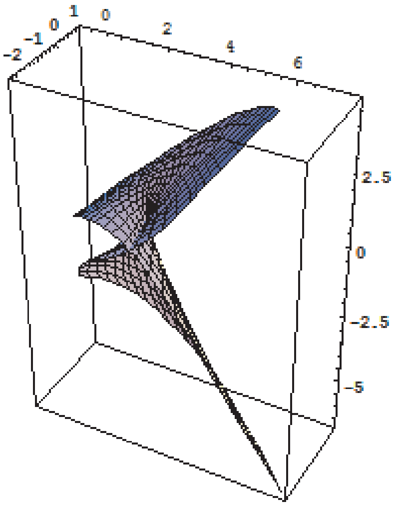
If we take $a = 2$, $b = 1$, $\beta = -1$, and $\gamma = 0$, then we immediately obtain a member of this family (see Fig. 1(a)). Fig. 1(b) shows another member in the family with $a = 2$, $b = 1$ and $\beta = 1$, $\gamma = 0$.

2 - By choosing $u(s,t) = 0$, $v(s,t) = 2t\tanh(s)$, $w(s,t) = t^2$, $t_0 = 0$, and $0 \le t \le 1$, then Eq. (17) is satisfied. Hence, we obtain a timelike surface pencil with a common spacelike asymptotic curve $\alpha(s)$ as

$$\begin{aligned} \mathbf{P}(s,t) &= \Big(a\sinh\frac{s}{c}, \frac{bs}{c}, a\cosh\frac{s}{c}\Big) \\ &+ (0, 2t\tanh(s), t^2)\begin{pmatrix} \frac{a}{c}\cosh\frac{s}{c} & \frac{b}{c} & \frac{a}{c}\sinh\frac{s}{c} \\ \sinh\frac{s}{c} & 0 & \cosh\frac{s}{c} \\ \frac{b}{c}\cosh\frac{s}{c} & -\frac{a}{c} & \frac{b}{c}\sinh\frac{s}{c} \end{pmatrix}. \end{aligned}$$



(a)



(b)

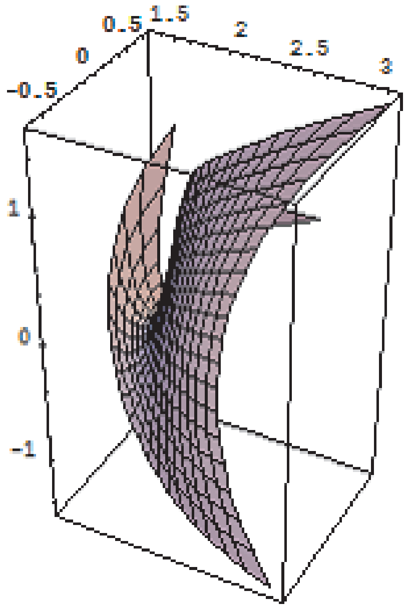
Fig. 1

If we take $a = 2$, and $b = 1$, then we immediately obtain a member of this family (see Fig. 2(a)). Fig. 2(b) shows another member in the family with $a = 2$, and $b = -1$.

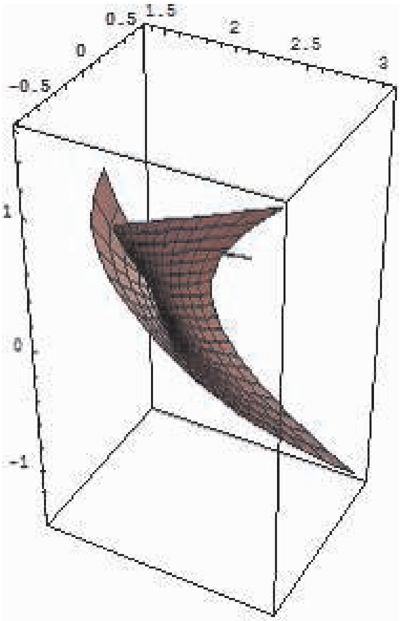
3 - By choosing $u(s,t) = \sinh\big(\sum_{k=1}^{4} s^k t^k\big)$, $v(s,t) = \sum_{k=1}^{4} s^k t^k$, $w(s,t) = 0$, $t_0 = 0$ and $-.5 \le t \le .5$, then Eq. (19) is satisfied. Hence, we obtain a timelike surface pencil with a common spacelike asymptotic curve $\alpha(s)$ as

$$\begin{aligned}\mathbf{P}(s,t) &= \Big(a\sinh\frac{s}{c}, \frac{bs}{c}, a\cosh\frac{s}{c}\Big) + \Big(\sinh\Big(\sum_{k=1}^{4} s^k t^k\Big), \sum_{k=1}^{4} s^k t^k, 0\Big)\\ &\quad\times\begin{pmatrix} \frac{a}{c}\cosh\frac{s}{c} & \frac{b}{c} & \frac{a}{c}\sinh\frac{s}{c}\\ \sinh\frac{s}{c} & 0 & \cosh\frac{s}{c}\\ \frac{b}{c}\cosh\frac{s}{c} & -\frac{a}{c} & \frac{b}{c}\sinh\frac{s}{c}\end{pmatrix}.\end{aligned}$$

If we take $a = 2$, and $b = 1$, then we immediately obtain a member of this family (see Fig. 3(a)). Fig. 3(b) shows another member in the family with $a = 2$, and $b = -1$.



(a)



(b)

Fig. 2

Example 3.2. Suppose we are given a parametric spacelike curve

$$\alpha(s) = (\sinh s, 0, \cosh s), \quad -1 \leq s \leq 1.$$

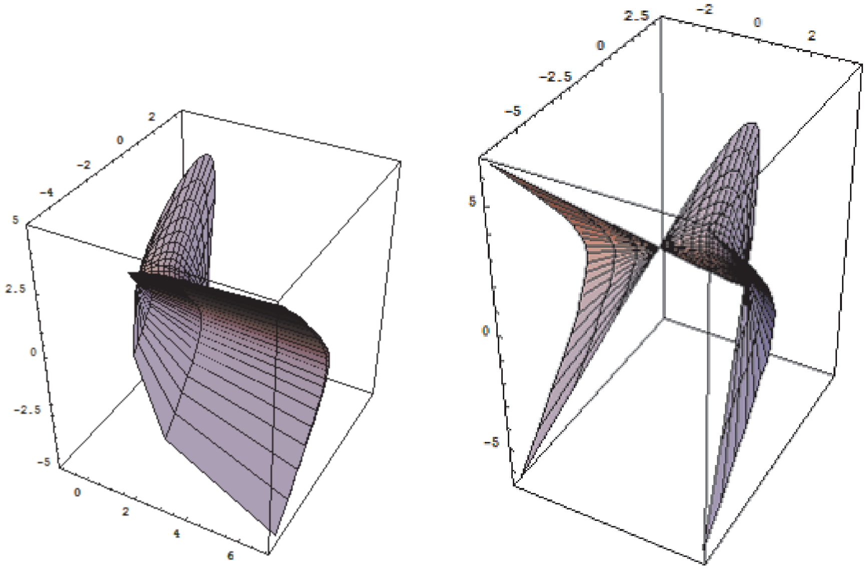
After simple computation, we have

$$\mathbf{t}(s) = (\cosh s, 0, \sinh s), \ \mathbf{n}(s) = (\sinh s, 0, \cosh s), \ \mathbf{b}(s) = (0, -1, 0).$$

If we choose $u(s,t) = 0$, $v(s,t) = \cos t + \sum_{k=2}^{p} a_{2k} \cos^k(t)$, $w(s,t) = 1 + \sin t + \sum_{k=2}^{p} a_{3k}(1 + \sin t)^k$, $t_0 = \dfrac{3\pi}{2}$ and $t \in [-3\pi/2, 3\pi/2]$, then Eq. (19) is satisfied. Thus, the timelike surfaces family with common spacelike asymptotic is given by

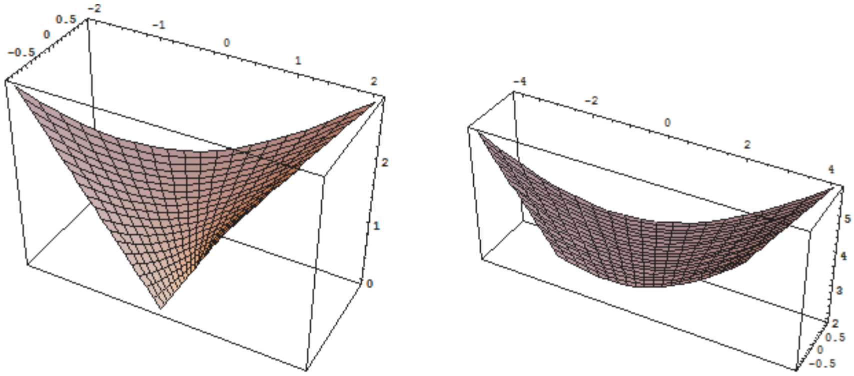
$$\mathbf{P}(s,t) = (\sinh s + v \sinh s, -w, \cosh s + v \cosh s).$$

If $\mathrm{a}_{2k} = 0.01$, and $\mathrm{a}_{3k} = 0.05$, then we immediately obtain a member of this family (Fig. 4(a)). If we take $\mathrm{a}_{2k} = 0$ and $a_{3k} = 0, k = 2, 3, 4$, then we obtain another member in this family (Fig. 4(b)).



(a) (b)

Fig. 3



(a) (b)

Fig. 4

3.2 - $\alpha = \alpha(s)$ *is a timelike asymptotic curve*

This time the directrix $\alpha(s)$ is a timelike asymptotic curve on the surface $\mathbf{P}(s,t)$. As stated in the above case, we can get the corresponding Theorem for Theorem 3.1; we omit the details here.

Theorem 3.2. *The given timelike curve* $\alpha(s)$ *is an asymptotic curve on* *the timelike surface* $\mathbf{P}(s,t)$ *if and only if*

$$
(20)\quad \left.
\begin{aligned}
&u(s,t_0)=v(s,t_0)=w(s,t_0)=0,\\
&\eta_1(s,t)=-\frac{\partial v(s,t)}{\partial s}\frac{\partial w(s,t)}{\partial t}+\frac{\partial w(s,t)}{\partial s}\frac{\partial v(s,t)}{\partial t}=0,\\
&\eta_2(s,t)=-\left(1+\frac{\partial u(s,t)}{\partial s}\right)\frac{\partial w(s,t)}{\partial t}+\frac{\partial w(s,t)}{\partial s}\frac{\partial u(s,t)}{\partial t}=0,\\
&\eta_3(s,t)=\left(1+\frac{\partial u(s,t)}{\partial s}\right)\frac{\partial v(s,t)}{\partial t}-\frac{\partial v(s,t)}{\partial s}\frac{\partial u(s,t)}{\partial t}\neq 0.
\end{aligned}
\right\}
$$

In addition, by a similar procedure, we have the same conditions in Eqs. (15), (17) and (19).

Example 3.3. Suppose we are given a parametric timelike curve

$$\alpha(s)=\left(a\cosh\frac{s}{c},\frac{bs}{c},a\sinh\frac{s}{c}\right),$$

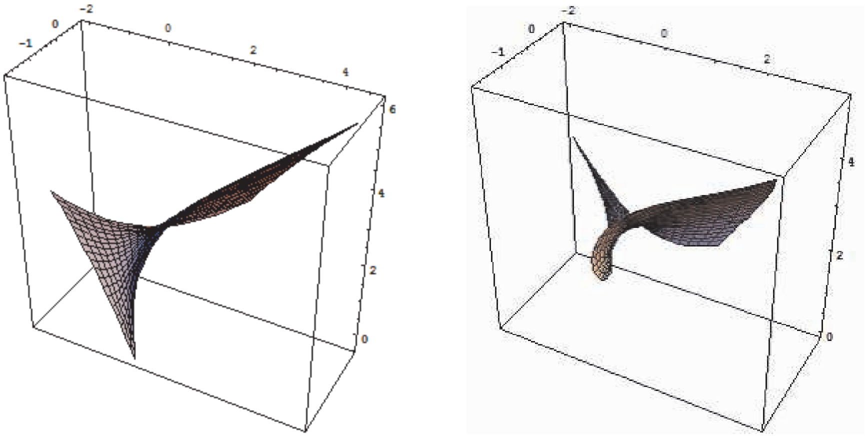
where $a,b,c\in\mathbb{R}$ $a^2-b^2=c^2$. We will construct a family of timelike surfaces sharing the curve $\alpha(s)$ as the timelike asymptotic curve.

1 - By choosing $u(s,t)=0,\ v(s,t)=\beta t,\ w(s,t)=\gamma t^2,$ where $\beta,\gamma\in\mathbb{R},\ \beta\neq 0,\ -3\leq s\leq 3,\ t_0=0,$ and $-2\leq t\leq 2$, then Eq. (15) is satisfied. Hence, we obtain the following timelike surface family with a common timelike asymptotic curve $\alpha(s)$ as

$$\mathbf{P}(s,t;\beta,\gamma)=(a\cosh\frac{s}{c},\frac{bs}{c},a\sinh\frac{s}{c})+t(0,\beta,\gamma t)\begin{pmatrix}\frac{a}{c}\sinh\frac{s}{c} & \frac{b}{c} & \frac{a}{c}\cosh\frac{s}{c}\\ \cosh\frac{s}{c} & 0 & \sinh\frac{s}{c}\\ \frac{b}{c}\sinh\frac{s}{c} & \frac{a}{c} & -\frac{b}{c}\cosh\frac{s}{c}\end{pmatrix}.$$

If we take $a=2,\ b=1,\ \beta=1,$ and $\gamma=-1$, then we immediately obtain a member of this family (see Fig. 5(a)). Fig. 5(b) shows another member in the family with $a=2,\ b=1,\ \beta=1,$ and $\gamma=0$.

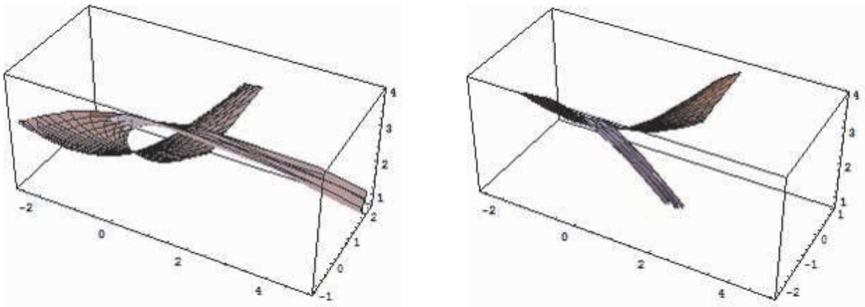
$$
\begin{aligned}
\mathbf{P}(s,t;\beta,\gamma) \;=\;& \left(a\cosh\frac{s}{c},\frac{bs}{c},a\sinh\frac{s}{c}\right)+(0,2t\tanh(s),t^2)\\
&\times\begin{pmatrix}\frac{a}{c}\sinh\frac{s}{c} & \frac{b}{c} & \frac{a}{c}\cosh\frac{s}{c}\\ \cosh\frac{s}{c} & 0 & \sinh\frac{s}{c}\\ \frac{b}{c}\sinh\frac{s}{c} & \frac{a}{c} & -\frac{b}{c}\cosh\frac{s}{c}\end{pmatrix},
\end{aligned}
$$



(a) (b)

Fig. 5

If we take $a = 2$, and $b = 1$, then we immediately obtain a member of this family (see Fig. 6(a)). Fig. 6(b) shows another member in the family with $a = 2$, and $b = -1$.



(a) (b)

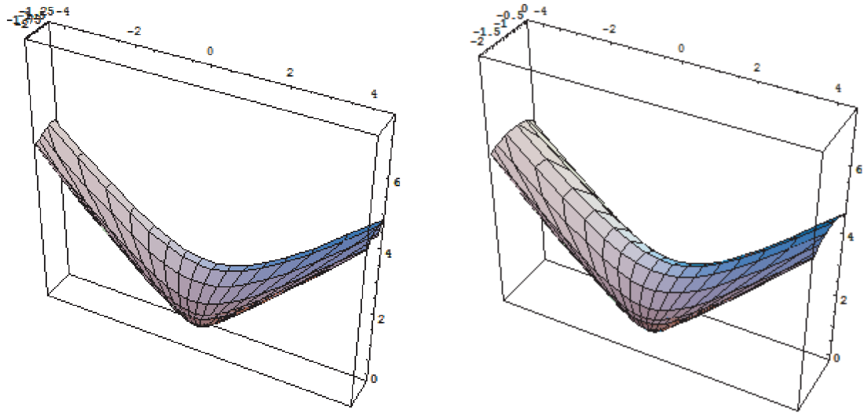
Fig. 6

3 - By choosing $u(s,t) = \sinh\left(\sum_{k=1}^{4} s^k t^k\right)$, $v(s,t) = \sum_{k=1}^{4} \cosh^k s \sinh^k t$, $w(s,t) = \sum_{k=2}^{4} \cos^k s \sin^k t$, $-2 \leq s \leq 2$, $t_0 = 0$, and $-1 \leq t \leq 1$, then Eq. (19) is satisfied. Hence, we obtain a timelike surface pencil with a common timelike asymptotic

curve $\alpha(s)$ as

$$\mathbf{P}(s,t) = \Big(a\cosh\frac{s}{c}, \frac{bs}{c}, a\sinh\frac{s}{c}\Big) + \Big(\sinh\big(\sum_{k=1}^{4} s^k t^k\big), \sum_{k=1}^{4} s^k t^k, 0\Big) \times \begin{pmatrix} \frac{a}{c}\sinh\frac{s}{c} & \frac{b}{c} & \frac{a}{c}\cosh\frac{s}{c} \\ \cosh\frac{s}{c} & 0 & \sinh\frac{s}{c} \\ \frac{b}{c}\sinh\frac{s}{c} & \frac{a}{c} & -\frac{b}{c}\cosh\frac{s}{c} \end{pmatrix}.$$

If we take $a = 2$, $b = 1$, then we immediately obtain a member of this family (see Fig. 7(a)). Fig. 7(b) shows another member in the family with $a = 2$, and $b = -1$.



(a) (b)

Fig. 7

Example 3.4. Suppose we are given a parametric timelike curve

$$\alpha(s) = (0, \cosh s, \sinh s), \quad -2 \le s \le 2.$$

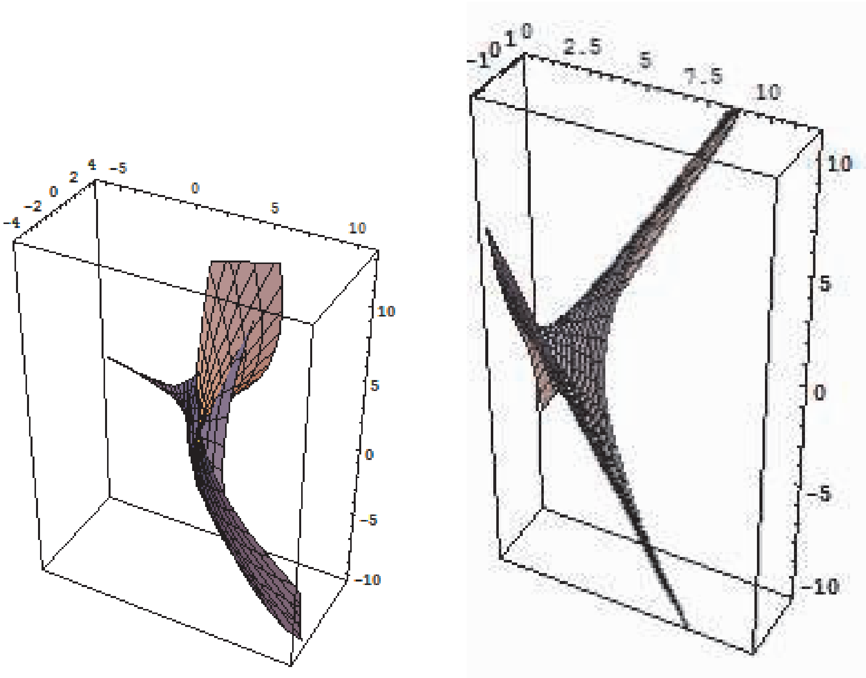
After simple computation, we have

$$\mathbf{t}(s) = (0, \sinh s, \cosh s),\ \mathbf{n}(s) = (0, \cosh s, \sinh s),\ \mathbf{b}(s) = (-1, 0, 0).$$

If we choose $u(s,t) = 0$, $v(s,t) = 1 - \cosh t + \sum_{k=2}^{4} a_{2k}(1 - \cosh t)^k$, $w(s,t) = \sinh t + \sum_{k=2}^{4} a_{3k}\sinh^k(t)$, where $t \in [0, 2\pi]$, then the timelike surfaces family with common timelike asymptotic is given by

$$\mathbf{P}(s,t) = (-w, \cosh s + v \cosh s, \sinh s + v \sinh s).$$

If $\mathrm{a}_{2k} = 0.01$ and $\mathrm{a}_{3k} = 0.05$, then we immediately obtain a member of this family (Fig. 8(a)). If we take $\mathrm{a}_{2k} = 0$ and $a_{3k} = 0, k = 2, 3, 4$, then we obtain another member in this family (Fig. 8(b)).



(a) (b)

Fig. 8

4 - Conclusion

In this study we have presented a method for finding a timelike surface family whose members all share a given asymptotic spacelike or timelike curve as an isoparametric curve. By representing the surface by the combination of the given curve, and the three vectors decomposed along the directions of Serret–Frenet, we derive the necessary and sufficient conditions for the given curve to be asymptotic for the parametric timelike surface. The members of timelike surfaces family with common asymptotic do not have the same properties. The member, having any desired property, of a surface family can be obtained by choosing the appropriate marching-scale functions $u(s,t)$, $v(s,t)$ and $w(s,t)$. There are several opportunities for further work. An analogue of the problem addressed in this paper may be consider for spacelike surfaces. Another possibility is to consider the geometric properties of the surfaces that are constructed?. For instance, what is the Gaussian curvature, the mean curvature, is it minimal, does it have constant mean curvature, . . .?. In addition, an analogue of the problem addressed in this paper may be considered for 3-surfaces in 4-space or other types of marching-scale functions may be investigated.

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