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Yûsuke Okuyama

Nevanlinna theory and value distribution in the unicritical polynomials family

Abstract. In the space $\mathbb{C}$ of the parameters λ of the unicritical polynomials family $f(\lambda, z) = f_\lambda(z) = z^d + \lambda$ of degree $d > 1$, we establish a quantitative equidistribution result towards the bifurcation current (indeed measure) T_f of f as $n \to \infty$ on the averaged distributions of all parameters λ such that f_λ has a superattracting periodic point of period n in $\mathbb{C}$, with a concrete error estimate for C^2-test functions on $\mathbb{P}^1$. In the proof, not only complex dynamics but also a standard argument from the Nevanlinna theory play key roles.

Keywords. unicritical polynomials family, superattracting periodic point, equidistribution, Nevanlinna theory.

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Contents

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1 - Introduction

Let $f : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{P}^1$ be the (*monic and centered*) *unicritical polynomials family*

$$f(\lambda, z) = f_\lambda(z) := z^d + \lambda \quad \text{for every } (\lambda, z) \in \mathbb{C} \times \mathbb{P}^1 \tag{1.1}$$

of degree $d > 1$. Let $c_0 \equiv 0$ on $\mathbb{C}$, which is a *marked critical point* of the family f in that for every $\lambda \in \mathbb{C}$, $c_0(\lambda)$ is a critical point of $f_\lambda(z) \in \mathbb{C}[z]$. For every $n \in \mathbb{N} \cup \{0\}$, let us define the monic polynomial

$$F_n(\lambda) := f_\lambda^n(c_0(\lambda)) \equiv f_\lambda^n(0) \in \mathbb{Z}[\lambda]$$

of degree d^{n-1}. Any zero of F_n is simple (Douady–Hubbard [**10**, Exposé XIX]; see also [**19**, Theorem 10.3] for a simple proof). The study of the asymptotic behavior as $n \to \infty$ of the set of all zeros of F_n, which is the set of all parameters $\lambda \in \mathbb{C}$ such that f_λ has a superattracting periodic point of (not necessarily exact) period n in $\mathbb{C}$, was initiated by Levin [**15**], and has been developed by Bassanelli–Berteloot [**2**, **3**] and Buff–Gauthier [**7**] subsequently.

Our aim is, from both complex dynamics and the *Nevanlinna theory*, to contribute to the quantitative study of the asymptotic behavior of zeros of F_n as $n \to \infty$, partly sharpening Gauthier–Vigny [**14**].

Notation 1.1. Let $\mu : \mathbb{N} \mapsto \{-1, 0, 1\}$ be the Möbius function from arithmetic (cf. [**1**, §2]). Let $\log^+ t := \log \max\{1, t\}$ on $\mathbb{R}$. Let ω be the Fubini-Study area element on $\mathbb{P}^1$ normalized as $\omega(\mathbb{P}^1) = 1$, let $[z, w]$ be the chordal metric on $\mathbb{P}^1$ normalized as $[\cdot, \infty] = 1/\sqrt{1 + |\cdot|^2}$ on $\mathbb{P}^1$ (following the notation in Nevanlinna's and Tsuji's books [**23**,**29**]), and let δ_x be the Dirac measure on $\mathbb{P}^1$ at each $x \in \mathbb{P}^1$. The Laplacian dd^c on $\mathbb{P}^1$ is normalized as $dd^c(-\log[\cdot, \infty]) = \omega - \delta_\infty$ on $\mathbb{P}^1$. Set $\mathbb{D}(x, r) := \{y \in \mathbb{C} : |x - y| < r\}$ for every $x \in \mathbb{C}$ and every $r > 0$, $\mathbb{D}(r) := \mathbb{D}(0, r)$ for every $r > 0$, and $\mathbb{D} := \mathbb{D}(1)$.

1.1 - *Main result*

Let $g_{I_{c_0}}$ be the Green function with pole ∞ on the *escaping* locus $I_{c_0} := \{\lambda \in \mathbb{C} : \limsup_{n \to \infty} |F_n(\lambda)| = \infty\}$ of the marked critical point c_0 of f; I_{c_0}

is a punctured open and connected neighborhood of ∞ in $\mathbb{P}^1$, and ∂I_{c_0} and $\mathbb{C} \setminus I_{c_0}$ respectively coincide with the *J-unstability* or *bifurcation* locus B_f and the *connectedness locus* M_f of f. The function $g_{I_{c_0}}$ extends to $\mathbb{C}$ continuously by setting $g_{I_{c_0}} \equiv 0$ on M_f, and $\mu_{B_f} := \mathrm{dd}^c g_{I_{c_0}} + \delta_\infty$ on $\mathbb{P}^1$ coincides with the harmonic measure on B_f with pole ∞. The *measure* $(d-1)d^{-1}\mu_{B_f}$ on $\mathbb{P}^1$ coincides with the *bifurcation current* (indeed measure) T_f of f on $\mathbb{P}^1$ (see Subsection 2.1). By a refinement of Przytycki's argument on the recurrence of critical orbits [**25**, Proof of Lemma 2] and Buff's upper estimate of the moduli of the derivatives of polynomials [**6**, the proof of Theorem 3], we will establish the following $L^1(\omega)$ estimate

$$\int_{\mathbb{P}^1} \big|\log|F_n| - d^{n-1} \cdot g_{I_{c_0}}\big|\omega \le \frac{2\log d}{d-1} n + O(1) \tag{1.2}$$

as $n \to \infty$, with the *concrete* coefficient $(2\log d)/(d-1)$ of n in the right hand side; a question on the best possibility of this estimate (1.2) seems also interesting. As seen in the proof of (1.2) (in Section 3), this may be regarded as a counterpart of H. Selberg's theorem [**26**, p. 313] from the Nevanlinna theory.

Our principal result is a deduction from (1.2) of the following *quantitative* equidistribution of the sequence $(F_n^*\delta_0/d^n)$ of the *averaged* distribution of the superattracting parameters of period n towards $(d-1)^{-1}T_f = d^{-1}\mu_{B_f}$ as $n \to \infty$.

Theorem 1. *Let $f : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{P}^1$ be the unicritical (monic and centered) polynomials family of degree $d > 1$ defined as in (1.1). Then for every $\phi \in C^2(\mathbb{P}^1)$,*

$$\left|\int_{\mathbb{P}^1} \phi \mathrm{d}\left((d-1)\cdot F_n^*\delta_0 - d^n \cdot T_f\right)\right| \le \left(\sup_{\mathbb{P}^1}\left|\frac{\mathrm{dd}^c\phi}{\omega}\right|\right) \cdot \big((2\log d)n + O(1)\big) \tag{1.3}$$

as $n \to \infty$, where the implicit constant in $O(1)$ is independent of ϕ and the Radon-Nikodim derivative $(\mathrm{dd}^c\phi)/\omega$ on $\mathbb{P}^1$ is bounded on $\mathbb{P}^1$.

For a former application of Selberg's theorem (Theorem 3.2) to obtain a quantitative equidistribution result in complex dynamics, see Drasin and the author [**12**]. As an order estimate, the estimate (1.3) is due to Gauthier–Vigny [**14**, Theorem A]. The implicit constant in $O(1)$ in (1.3) will also be computed in the proof. The coefficient $2\log d$ of n in (1.3) comes from the full strength of de Branges's theorem (the solution of the Bieberbach conjecture), on which the proof of Buff's estimate mentioned above essentially relies.

1.2 - *Non-repelling parameters having exact periods*

For every $n \in \mathbb{N}$, the n-th *dynatomic polynomial*

$$\Phi^*_{f,n}(\lambda, z) := \prod_{m \in \mathbb{N}:\, m|n} (f^m_\lambda(z) - z)^{\mu(n/m)}$$

of the family f is in fact in $\mathbb{Z}[\lambda, z]$, and for every $\lambda \in \mathbb{C}$, $\Phi^*_{f,n}(\lambda, z) \in \mathbb{C}[z]$ is *monic* and of degree

$$\nu(n) = \nu_d(n) := \sum_{m \in \mathbb{N}:\, m|n} \mu\left(\frac{n}{m}\right) d^m. \tag{1.4}$$

For every $\lambda \in \mathbb{C}$ and every $n \in \mathbb{N}$, let $\mathrm{Fix}_f(\lambda, n)$ be the set of all fixed points of f^n_λ in $\mathbb{C}$ and set $\mathrm{Fix}^*_f(\lambda, n) := \mathrm{Fix}_f(\lambda, n) \setminus \left(\bigcup_{m \in \mathbb{N}:\, m|n \text{ and } m<n} \mathrm{Fix}_f(\lambda, m)\right)$, each element in which is called a periodic point of f_λ in $\mathbb{C}$ having the *exact* period n. For every $n \in \mathbb{N}$ and every $\lambda \in \mathbb{C}$, a periodic point z of f_λ in $\mathbb{C}$ is said to have the *formally exact* period n if either (i) $z \in \mathrm{Fix}^*_f(\lambda, n)$ or (ii) there is an $m \in \mathbb{N}$ satisfying $m|n$ and $m < n$ such that $z \in \mathrm{Fix}^*_f(\lambda, m)$ and that $(f^m_\lambda)'(z)$ is a primitive (n/m)-th root of unity (so in particular $(f^n_\lambda)'(z) = 1$). For every $\lambda \in \mathbb{C}$ and every $n \in \mathbb{N}$, let $\mathrm{Fix}^{**}_f(\lambda, n)$ be the set of all periodic points of f_λ in $\mathbb{C}$ having the formally exact period n, which in fact coincides with $(\Phi^*_{f,n}(\lambda, \cdot))^{-1}(0)$. For every $n \in \mathbb{N}$, the n-th *multiplier polynomial*

$$p^*_{f,n}(\lambda, w) := \left(\prod_{z \in \mathrm{Fix}^{**}_f(\lambda, n)} ((f^n_\lambda)'(z) - w)\right)^{1/n}$$

of f, where for each $\lambda \in \mathbb{C}$, the product in the right hand side takes into account the multiplicity of each $z \in \mathrm{Fix}^{**}_f(\lambda, n)$ as a zero of $\Phi^*_{f,n}(\lambda, \cdot)$, is indeed in $\mathbb{Z}[\lambda, w]$ and unique up to multiplication in n-th roots of unity. For every $w \in \mathbb{C}$, by a direct computation,

$$\deg_\lambda p^*_{f,n}(\lambda, w) = (d-1)\frac{\nu(n)}{d} \tag{1.5}$$

and the coefficient of the leading term of $p^*_{f,n}(\lambda, w) \in \mathbb{C}[\lambda]$ equals $d^{\nu(n)}$, both of which are independent of w. For every $n \in \mathbb{N}$ and every $w \in \mathbb{C}$, let $\mathrm{Per}^*_f(n, w)$ be the effective divisor on $\mathbb{P}^1$ defined by the zeros of $p^*_{f,n}(\lambda, w) \in \mathbb{C}[\lambda]$; as a Radon measure on $\mathbb{P}^1$,

$$\mathrm{Per}^*_f(n, w) = dd^c_\lambda \log |p^*_{f,n}(\lambda, w)| + (d-1)\frac{\nu(n)}{d}\delta_\infty.$$

For more details, see e.g. [**28**, §4], [**4**, §2.3], [**21**, §3].

Notation 1.2. Let $(\sigma_0(n))$ and $(\sigma_1(n))$ be such sequences in $\mathbb{N}$ that $1 = \sum_{m\in\mathbb{N}:\, m|n}\mu(n/m)\sigma_0(m)$ and $n = \sum_{m\in\mathbb{N}:\, m|n}\mu(n/m)\sigma_1(m)$, or equivalently, $\sigma_0(n) = \sum_{m\in\mathbb{N}:\, m|n} 1$ and $\sigma_1(n) = \sum_{m\in\mathbb{N}:\, m|n} m$ by Möbius inversion, for every $n \in \mathbb{N}$.

By an argument similar to that in the proof of Theorem 1, we will also show the following.

Theorem 2. *Let $f : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{P}^1$ be the unicritical (monic and centered) polynomials family of degree $d > 1$ defined as in (1.1). Then for every $\phi \in C^2(\mathbb{P}^1)$,*

$$\left|\int_{\mathbb{P}^1} \phi \mathrm{d}\left(\mathrm{Per}^*_f(n,0) - \nu(n)\cdot T_f\right)\right| \leq \left(\sup_{\mathbb{P}^1}\left|\frac{\mathrm{dd}^c\phi}{\omega}\right|\right)\cdot\left((2\log d)\sigma_1(n) + O(\sigma_0(n))\right) \tag{1.6}$$

as $n \to \infty$, where the term $O(\sigma_0(n))$ is independent of ϕ, and for every $\phi \in C^2(\mathbb{P}^1)$ and every $r \in (0, 1]$,

$$\left|\int_{\mathbb{P}^1} \phi \mathrm{d}\left(\int_0^{2\pi} \mathrm{Per}^*_f(n, re^{i\theta})\frac{\mathrm{d}\theta}{2\pi} - \nu(n)\cdot T_f\right)\right| \leq \left(\sup_{\mathbb{P}^1}\left|\frac{\mathrm{dd}^c\phi}{\omega}\right|\right)\cdot\left((2\log d)\sigma_1(n) + O(\sigma_0(n))\right) \tag{1.7}$$

as $n \to \infty$, where the term $O(\sigma_0(n))$ is independent of both ϕ and r. Here the Radon-Nikodim derivative $(\mathrm{dd}^c\phi)/\omega$ on $\mathbb{P}^1$ is bounded on $\mathbb{P}^1$.

Again, the terms $O(\sigma_0(n))$ in Theorem 2 will also be computed in Section 4. As an order estimate, the estimate (1.6) is a consequence of Gauthier–Vigny [**14**, Theorem A]. The estimate (1.7) quantifies Bassanelli–Berteloot [**3**, 2. in Theorem 3.1] for $r \in (0, 1]$.

1.3 - *Organization of the article*

In Section 2, we recall background from the study of the unicritical polynomials family f. In Section 3, we show Theorem 1. In Section 4, we show Theorem 2.

2 - Background from the study of the family f

Let $f : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{P}^1$ be the unicritical (monic and centered) polynomials family of degree $d > 1$ defined as in (1.1), and recall that $c_0(\lambda) = 0 \in \mathbb{Z}[\lambda]$ defines a marked critical point of f.

2.1 - *Douady–Hubbard's theory on the parameter space* $\mathbb{C}$ *of* f

For every $\lambda \in \mathbb{C}$, let J_{f_λ} be the Julia set of f_λ, which is compact in $\mathbb{C}$. Let B_f be the *J-unstability* or *bifurcation* locus of the family f, which is the discontinuity locus of the set function $\lambda \mapsto J_{f_\lambda}$ with respect to the Hausdorff topology from $(\mathbb{P}^1, [z, w])$, and is closed and nowhere dense in $\mathbb{C}$ (by Mañé–Sad–Sullivan [**17**], Lyubich [**16**]). The *escaping* locus

$$I_{c_0} := \{\lambda \in \mathbb{C} : \limsup_{n\to\infty} |F_n(\lambda)| = \infty\}$$

of the marked critical point c_0 of f is a punctured open and connected neighborhood of ∞ in $\mathbb{P}^1$ and coincides with the unique unbounded component of $\mathbb{C} \setminus B_f$. We have $B_f = \partial I_{c_0}$, and the *connectedness* locus

$$M_f := \{\lambda \in \mathbb{C} : J_{f_\lambda} \text{ is connected}\}$$

of f coincides with $\mathbb{C} \setminus I_{c_0}$ (and is connected). For every $\lambda \in \mathbb{C}$, f_λ has at most one non-repelling cycle in $\mathbb{C}$ (see, e.g., [**20**, §8]). Let H_f be the *hyperbolicity* locus of f, which coincides with the union of I_{c_0} and the set of all $\lambda \in M_f$ such that f_λ has the (super)attracting cycle in $\mathbb{C}$, and is a closed and open subset in $\mathbb{C} \setminus B_f$. For example, for every $n \in \mathbb{N}$, $0 \in F_n^{-1}(0) \subset H_f \setminus I_{c_0}$. For every component U of $H_f \setminus I_{c_0}$, there are an $n_U \in \mathbb{N}$ and a proper holomorphic mapping $\phi_U : U \to \mathbb{D}$ of degree $d - 1$ such that $\#\phi_U^{-1}(0) = 1$ and that for every $w \in \mathbb{D}$, $\phi_U^{-1}(w)$ coincides with the set of all $\lambda \in U$ such that f_λ has the (super)attracting cycle in $\mathbb{C}$ having the *exact* period n_U and the multiplier w. For more details, see Douady–Hubbard [**11**], and for a modern treatment, see McMullen–Sullivan [**19**, §10].

2.2 - *The Green functions on the dynamical and parameter spaces*

For every $\lambda \in \mathbb{C}$, J_{f_λ} coincides with the boundary of the filled-in Julia set $K_{f_\lambda} := \{z \in \mathbb{C} : \limsup_{n\to\infty} |f_\lambda^n(z)| < \infty\}$ of f_λ, which is compact in $\mathbb{C}$. For every $\lambda \in \mathbb{C}$, the uniform limit

$$g_{f_\lambda}(z) := \lim_{n\to\infty} \frac{-\log[f_\lambda^n(z), \infty]}{d^n} \tag{2.1}$$

exists on $\mathbb{C}$, and setting $g_{f_\lambda}(\infty) := +\infty$, the probability measure $\mu_{f_\lambda} := dd^c g_{f_\lambda} + \delta_\infty$ on $\mathbb{P}^1$ coincides with the harmonic measure on J_{f_λ} with pole ∞. Moreover, μ_{f_λ} is mixing so ergodic under f_λ (by Brolin [**5**]). For completeness, we include a proof of the following.

Lemma 2.1. *For every $\lambda \in \mathbb{C}$,*

$$\sup_{\mathbb{C}} \left| g_{f_\lambda} + \log[\cdot, \infty] \right| \le \frac{1}{d-1} \cdot \sup_{z \in \mathbb{C}} \left| \log \frac{[z,\infty]^d}{[f_\lambda(z),\infty]} \right|, \tag{2.2}$$

and the function $\lambda \mapsto \sup_{z\in\mathbb{C}} |\log([z,\infty]^d/[f_\lambda(z),\infty])|$ is locally bounded on $\mathbb{C}$.

Proof. For every $\lambda \in \mathbb{C}$, by the definition (2.1) of g_{f_λ}, we have

$$\begin{aligned}\sup_{\mathbb{C}} \left| g_{f_\lambda} + \log[\cdot, \infty] \right| &\le \sup_{z\in\mathbb{C}} \left| \sum_{j=1}^{\infty} \frac{-\log[f_\lambda(f_\lambda^{j-1}(z)),\infty] + d\cdot \log[f_\lambda^{j-1}(z),\infty]}{d^j} \right| \\ &\le \frac{1}{d-1} \cdot \sup_{z\in\mathbb{C}} \left| \log \frac{[z,\infty]^d}{[f_\lambda(z),\infty]} \right|.\end{aligned}$$

For every $\lambda \in \mathbb{C}$, let us define the non-degenerate homogeneous polynomial endomorphism $\tilde{f}_\lambda : \mathbb{C}^2 \to \mathbb{C}^2$ of degree d by $\tilde{f}_\lambda(p_0,p_1) := (p_0^d, p_0^d f_\lambda(p_1/p_0)) = (p_0^d, p_1^d + \lambda p_0^d)$. Then the function $(\lambda,(p_0,p_1)) \mapsto \left|\log \|\tilde{f}_\lambda(p_0,p_1)\|\right|$ is continuous on $\mathbb{C} \times (\mathbb{C}^2 \setminus \{(0,0)\})$, and for every compact subset K in $\mathbb{C}$, we have

$$\sup_{(\lambda,z)\in K\times\mathbb{C}} \left| \log \frac{[z,\infty]^d}{[f_\lambda(z),\infty]} \right| = \sup_{(\lambda,(p_0,p_1))\in K\times S(1)} \left|\log \|\tilde{f}_\lambda(p_0,p_1)\|\right|,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{C}^2$ and $S(1) := \{(p_0,p_1) \in \mathbb{C}^2 : \|(p_0,p_1)\| = 1\}$. Now the proof is complete by the compactness of K in $\mathbb{C}$ and that of $S(1)$ in $\mathbb{C}^2 \setminus \{(0,0)\}$. $\square$

Similarly, the locally uniform limit

$$\lambda \mapsto g_{I_{c_0}}(\lambda) := \lim_{n\to\infty} \frac{-\log[F_n(\lambda),\infty]}{d^{n-1}} = d \cdot g_{f_\lambda}(c_0(\lambda)) = g_{f_\lambda}(f_\lambda(c_0(\lambda)))$$

exists on $\mathbb{C}$, and setting $g_{I_{c_0}} := +\infty$, the probability measure

$$\mu_f := dd^c g_{I_{c_0}} + \delta_\infty \quad \text{on } \mathbb{P}^1$$

coincides with the harmonic measure on $B_f = \partial I_{c_0}$ with pole ∞ (by Douady–Hubbard [**11**], Sibony [**27**]). The *activity current* (indeed measure) of the marked critical point c_0 of f is

$$T_{c_0} := \lim_{n\to\infty} \frac{F_n^*\omega}{d^n} = \frac{\mu_f}{d}$$

as currents on $\mathbb{P}^1$ (DeMarco [**8**], Dujardin–Favre [**13**]). For every $\lambda \in \mathbb{C}$, the *Lyapunov exponent* of f_λ with respect to μ_{f_λ} is

$$L(f_\lambda) := \int_{\mathbb{P}^1} \log |f'_\lambda(z)| \mathrm{d}\mu_{f_\lambda}(z) = \log d + (d-1)\frac{g_{I_{c_0}}}{d} (\geq \log d > 0)$$

(Manning [**18**], Przytycki [**24**]). Setting $L(f_\lambda)|_{\lambda=\infty} := +\infty$, the *bifurcation current* of f can be defined by

$$T_f := \mathrm{dd}^c L(f_\cdot) + \frac{d-1}{d}\delta_\infty = (d-1)\frac{\mu_f}{d} = (d-1)T_{c_0} \quad \text{on } \mathbb{P}^1 \tag{2.3}$$

(DeMarco [**9**]). For more details, see, e.g., Berteloot's survey [**4**, §3.2.3].

3 - Proof of Theorem 1

Let $f : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{P}^1$ be the unicritical polynomials family of degree $d > 1$ defined as (1.1). For every $\lambda \in \mathbb{C}$ and every $n \in \mathbb{N}$, let us define the *chordal derivative*

$$(f_\lambda^n)^\# := \sqrt{\frac{(f_\lambda^n)^*\omega}{\omega}} : \mathbb{P}^1 \to \mathbb{R}_{\geq 0}$$

of f_λ^n on $\mathbb{P}^1$. For every non-empty subset S in $\mathbb{P}^1$, let $\mathrm{diam}_\#(S)$ be the chordal diameter of S. The resultant of $(P(z), Q(z)) \in \mathbb{C}[z] \times \mathbb{C}[z]$ is denoted by $\mathrm{Res}(P, Q)$, as usual. Recall that $\{z \in \mathbb{C} : [z, 0] < [r, 0]\} = \mathbb{D}(0, r)$ for every $r > 0$ and that $[z, w] \leq |z - w|$ on $\mathbb{C} \times \mathbb{C}$.

Lemma 3.1. *For every $n \in \mathbb{N}$ and every $\lambda \in \mathbb{C} \setminus (H_f \setminus I_{c_0})$ (so in particular for every $\lambda \in B_f$),*

$$|F_n(\lambda)| \geq (\sqrt{2} - 1)\Big(2^{d+1} \cdot \sup_{z \in \mathbb{P}^1}((f_\lambda^{n-1})^\#(z))\Big)^{-1/(d-1)}.$$

Proof. Fix $n \in \mathbb{N}$ and define the functions L_{n-1} and ϵ_n on $\mathbb{C}$ by $L_{n-1}(\lambda) := \sup_{z \in \mathbb{P}^1}((f_\lambda^{n-1})^\#(z)) (> 1)$ and $\epsilon_n(\lambda) := (2^2 \cdot L_{n-1}(\lambda))^{-1/(d-1)} (< 1)$. For every $\lambda \in \mathbb{C}$, noting that $f_\lambda(0) = \lambda$ and that $f_\lambda(z) - f_\lambda(0) = z^d$ on $\mathbb{C}$, we have

$$\begin{aligned}
\mathrm{diam}_\#\big(f_\lambda^n(\{z \in \mathbb{C} : [z, 0] < [\epsilon_n(\lambda), 0]\})\big) &= \mathrm{diam}_\#\big(f_\lambda^n(\mathbb{D}(0, \epsilon_n(\lambda)))\big) \\
&= \mathrm{diam}_\#\big(f_\lambda^{n-1}(\mathbb{D}(\lambda, \epsilon_n(\lambda)^d))\big) \\
&\leq L_{n-1}(\lambda) \cdot \mathrm{diam}_\#(\mathbb{D}(\lambda, \epsilon_n(\lambda)^d)) \leq L_{n-1}(\lambda) \cdot 2\epsilon_n(\lambda)^d = \frac{\epsilon_n(\lambda)}{2},
\end{aligned}$$

so that if $[f_\lambda^n(0),0] < [\epsilon_n(\lambda),0] - \epsilon_n(\lambda)/2$, then $\sup\{[w,0] : w \in f_\lambda^n(\{z \in \mathbb{C} : [z,0] \le [\epsilon_n(\lambda),0]\})\} < ([\epsilon_n(\lambda),0] - \epsilon_n(\lambda)/2) + \epsilon_n(\lambda)/2 = [\epsilon_n(\lambda),0]$, i.e., $f_\lambda^n(\{z \in \mathbb{C} : [z,0] < [\epsilon_n(\lambda),0]\}) \Subset \{z \in \mathbb{C} : [z,0] < [\epsilon_n(\lambda),0]\}$; then by Brouwer's fixed point theorem, Montel's theorem, and Fatou's classification of cyclic Fatou components (see e.g. [**20**, §16]), the domain $\{z \in \mathbb{C} : [z,0] < [\epsilon_n(\lambda),0]\}$, which contains both the critical point $c_0(\lambda)(=0)$ of f_λ and a fixed point of f_λ^n, is contained in the immediate basin of a (super)attracting cycle of f_λ in $\mathbb{C}$.

Hence for every $\lambda \in \mathbb{C}$, we obtain the desired lower estimate

$$\begin{aligned}|F_n(\lambda)| \ge ([F_n(\lambda),0] =)[f_\lambda^n(0),0] &\ge [\epsilon_n(\lambda),0] - \frac{\epsilon_n(\lambda)}{2}\\ &\ge \left(\sqrt{2}-1\right)\frac{\epsilon_n(\lambda)}{2} = \left(\sqrt{2}-1\right)(2^{d+1}L_{n-1}(\lambda))^{-1/(d-1)}\end{aligned}$$

of $|F_n(\lambda)|$ unless 0 is in the immediate basin of a (super)attracting cycle of f_λ in $\mathbb{C}$. Now the proof is complete. □

The following is substantially shown in Buff [**6**, the proof of Theorem 4].

Theorem 3.1 (Buff). *Let $f \in \mathbb{C}[z]$ be of degree $d > 1$, and let $z_0 \in \mathbb{C}$. If $g_f(z_0) \ge \max_{c \in C(f)\cap\mathbb{C}} g_f(c)$, where g_f is the Green function of the filled-in Julia set K_f of f with pole ∞ and $C(f)$ is the set of all critical points of f, then $|f'(z_0)| \le d^2 \cdot e^{(d-1)g_f(z_0)}$, and the equality never holds if $C(f) \cap \mathbb{C}$ is not contained in K_f.*

Lemma 3.2. *For every $n \in \mathbb{N}$ and every $\lambda \in M_f$,*

$$\log\left(\sup_{z\in\mathbb{P}^1}((f_\lambda^n)^\#(z))\right) \le (2\log d)n + \frac{4}{d-1}\cdot\sup_{z\in\mathbb{C}}\left|\log\frac{[z,\infty]^d}{[f_\lambda(z),\infty]}\right|.$$

Proof. For every $n \in \mathbb{N}$, every $\lambda \in M_f$, and every $z \in \mathbb{C}$, by Theorem 3.1, we have $|(f_\lambda^n)'(z)| \le (d^n)^2 e^{(d^n-1)g_{f_\lambda}(z)}$, and by the definition (2.1) of g_{f_λ}, we have $0 \le (d^n-1)g_{f_\lambda}(z) = g_{f_\lambda}(f_\lambda^n(z)) - g_{f_\lambda}(z)$, so that

$$\begin{aligned}(f_\lambda^n)^\#(z) =&|(f_\lambda^n)'(z)|\cdot\frac{[f_\lambda^n(z),\infty]^2}{[z,\infty]^2}\\ \le& d^{2n}e^{g_{f_\lambda}(f_\lambda^n(z))-g_{f_\lambda}(z)}\cdot e^{2(\log[f_\lambda^n(z),\infty]-\log[z,\infty])}\\ \le& d^{2n}\cdot e^{2(g_{f_\lambda}(f_\lambda^n(z))+\log[f_\lambda^n(z),\infty])-2(g_{f_\lambda}(z)+\log[z,\infty])}\\ \le& d^{2n}\cdot e^{4\sup_{\mathbb{C}}|g_{f_\lambda}+\log[\cdot,\infty]|}.\end{aligned}$$

This with (2.2) completes the proof. □

Recalling the latter half of Lemma 2.1, we can set

$$C_{B_f} := \sup_{(\lambda,z)\in B_f\times\mathbb{C}} \left|\log\frac{[z,\infty]^d}{[f_\lambda(z),\infty]}\right| < \infty.$$

Then for every $n\in\mathbb{N}$, by Lemmas 3.1 and 3.2, we have

$$\inf_{B_f}\log|F_n| \ge -\frac{1}{d-1}\Big((d+1)\log 2 + (2\log d)(n-1) + \frac{4C_{B_f}}{d-1}\Big) + \log\big(\sqrt{2}-1\big).$$

On the other hand, for every $n\in\mathbb{N}$ and every $\lambda\in M_f$, by Buff [**6**, Theorem 1], we also have $F_n(\lambda) = f_\lambda^n(c_0(\lambda)) \in K_{f_\lambda} \subset \mathbb{D}(2)$. Hence for every $n\in\mathbb{N}$, we have the following uniform estimate

$$\begin{aligned}(3.1)\quad &\sup_{B_f}\big|\log|F_n|\big| \\ &\le \frac{1}{d-1}\Big((d+1)\log 2 + (2\log d)(n-1) + \frac{4C_{B_f}}{d-1} + (d-1)\log\big(\sqrt{2}+1\big)\Big) =: t_n.\end{aligned}$$

Now let us recall the following classical theorem from the Nevanlinna theory; for a modern formulation, see [**30**].

Theorem 3.2 (Selberg [**26**, p. 311]). *Let V be a bounded and at most finitely connected domain in $\mathbb{C}$ whose boundary components are piecewise real analytic Jordan closed curves, so that for every $y\in V$, the Green function $G_V(\cdot,y)$ on V with pole y exists and extends continuously to $\mathbb{C}$ by setting $\equiv 0$ on $\mathbb{C}\setminus V$. If V is in $\mathbb{C}\setminus\{0\}$, then for every $y\in V$ and every $r>0$, setting $\theta_V(r) := \int_{\{\theta\in[0,2\pi]:\, re^{i\theta}\in V\}} \mathrm{d}\theta \in [0,2\pi]$, we have*

$$(3.2)\qquad \int_0^{2\pi} G_V(re^{i\theta},y)\frac{\mathrm{d}\theta}{2\pi} \le \min\left\{\frac{\pi}{2}\tan\frac{\theta_V(r)}{4}, \log^+\frac{r}{\inf_{z\in V}|z|}\right\}.$$

Let H_1 be the component of H_f containing 0 and set

$$C_0 := \pi + \int_0^\infty \frac{2r}{(1+r^2)^2}\log^+\frac{r}{\sup\{t>0:\mathbb{D}(t)\subset H_1\}}\,\mathrm{d}r + \int_{H_1} G_{H_1}(\cdot,0)\omega < \infty.$$

Fix $n\in\mathbb{N}$. Recall that $\deg F_n = d^{n-1}$.

Claim 1.

$$\int_{F_n^{-1}(\mathbb{D}(e^{-t_n}))} \big|\log|F_n| - d^{n-1}\cdot g_{I_{c_0}}\big|\omega \le \omega(F_n^{-1}(\mathbb{D}(e^{-t_n})))t_n + C_0.$$

Proof. By (3.1), we have $\inf_{B_f} |F_n| \geq e^{-t_n}$. Let $\mathcal{F}$ be the family of all components of $F_n^{-1}(\mathbb{D}(e^{-t_n}))$, so that $\#\mathcal{F} \leq d^{n-1}$. By the description of H_f in Subsection 2.1, every $V \in \mathcal{F}$ is a piecewise real analytic Jordan domain in $H_f \setminus I_{c_0}$ and, since any zero of F_n is also simple, for every $V \in \mathcal{F}$, the restriction $F_n|V : V \to \mathbb{D}(e^{-t_n})$ is conformal. For every $V \in \mathcal{F}$, set $\lambda_V := (F_n|V)^{-1}(0)$. Let V_0 be the element of $\mathcal{F}$ containing 0. Recall the notation in Theorem 3.2. For every $V \in \mathcal{F}$, by the conformal invariance of the Green functions, we have

$$\log \frac{e^{-t_n}}{|F_n|} = G_{\mathbb{D}(e^{-t_n})}(F_n, 0) = G_V(\cdot, \lambda_V) \quad \text{on } V.$$

For every $r > 0$, fixing such $V_r \in \mathcal{F} \setminus \{V_0\}$ that for every $V \in \mathcal{F} \setminus \{V_0\}$, $\theta_{V_r}(r) \geq \theta_V(r)$ (so in particular that for every $V \in \mathcal{F} \setminus \{V_0, V_r\}$, $\theta_V(r) \in [0, \pi]$ since $2\pi \geq \theta_{V_r}(r) + \theta_V(r) \geq 2\theta_V(r) \geq 0$), we have

$$\begin{aligned}
&\sum_{V \in \mathcal{F}} \int_0^{2\pi} G_V(re^{i\theta}, \lambda_V) \frac{d\theta}{2\pi} \\
&= \sum_{V \in \mathcal{F} \setminus \{V_0\}} \int_0^{2\pi} G_V(re^{i\theta}, \lambda_V) \frac{d\theta}{2\pi} + \int_0^{2\pi} G_{V_0}(re^{i\theta}, 0) \frac{d\theta}{2\pi} \\
&\leq \Bigg(\sum_{V \in \mathcal{F} \setminus \{V_0, V_r\}} \Big(\frac{\pi}{2} \tan \frac{\theta_V(r)}{4} \Big) + \log^+ \frac{r}{\inf_{z \in V_r} |z|} \Bigg) + \int_0^{2\pi} G_{H_1}(re^{i\theta}, 0) \frac{d\theta}{2\pi} \\
&\leq \frac{\pi}{2} \cdot \sum_{V \in \mathcal{F} \setminus \{V_0, V_r\}} \frac{\theta_V(r)}{\pi} + \log^+ \frac{r}{\sup\{t > 0 : \mathbb{D}(t) \subset H_1\}} + \int_0^{2\pi} G_{H_1}(re^{i\theta}, 0) \frac{d\theta}{2\pi} \\
&\leq \frac{\pi}{2} \cdot \frac{2\pi}{\pi} + \log^+ \frac{r}{\sup\{t > 0 : \mathbb{D}(t) \subset H_1\}} + \int_0^{2\pi} G_{H_1}(re^{i\theta}, 0) \frac{d\theta}{2\pi},
\end{aligned}$$

where the first inequality is by (3.2) and the monotonicity of the Green functions, and the second inequality is by $\theta_V(r) \in [0, \pi]$ for every $V \in \mathcal{F} \setminus \{V_0, V_r\}$. Hence, since $t_n \geq 0$, we have

$$\begin{aligned}
\int_{F_n^{-1}(\mathbb{D}(e^{-t_n}))} \big|\log |F_n|\big| \omega &= \int_{F_n^{-1}(\mathbb{D}(e^{-t_n}))} (-\log |F_n|) \omega \\
&= \omega(F_n^{-1}(\mathbb{D}(e^{-t_n}))) t_n + \int_0^{\infty} \frac{2r dr}{(1 + r^2)^2} \sum_{V \in \mathcal{F}} \int_0^{2\pi} G_V(re^{i\theta}, \lambda_V) \frac{d\theta}{2\pi} \\
&\leq \omega(F_n^{-1}(\mathbb{D}(e^{-t_n}))) t_n + C_0,
\end{aligned}$$

which completes the proof. □

Claim 2. $\sup_{\mathbb{C}\setminus F_n^{-1}(\mathbb{D}(e^{-t_n}))}\left|\log|F_n| - d^{n-1}\cdot g_{I_{c_0}}\right| \le t_n$.

Proof. By the description of H_f in Subsection 2.1, the function $\log|F_n| - d^{n-1}\cdot g_{I_{c_0}}$ is not only harmonic on I_{c_0} but also bounded around ∞ so, by the removable singularity theorem for subharmonic functions twice, extends *harmonically* to $I_{c_0}\cup\{\infty\}$. Applying the maximum principle to this harmonic extension on $I_{c_0}\cup\{\infty\}$ twice, by $g_{I_{c_0}}\equiv 0$ on M_f and (3.1), we have $\sup_{I_{c_0}}\left|\log|F_n| - d^{n-1}\cdot g_{I_{c_0}}\right| \le \sup_{B_f}\left|\log|F_n|\right| \le t_n$ (cf. [**14**, the proof of Lemma 4.1]). Similarly, applying the maximum principle twice to the restriction of $\log|F_n|$ on $M_f\setminus F_n^{-1}(\mathbb{D}(e^{-t_n}))$, which is harmonic on the interior of $M_f\setminus F_n^{-1}(\mathbb{D}(e^{-t_n}))$, by $g_{I_{c_0}}\equiv 0$ on M_f and (3.1), we have $\sup_{M_f\setminus F_n^{-1}(\mathbb{D}(e^{-t_n}))}\left|\log|F_n| - d^{n-1}\cdot g_{I_{c_0}}\right| \le \sup_{B_f\cup F_n^{-1}(\partial\mathbb{D}(e^{-t_n}))}\left|\log|F_n|\right| \le t_n$. Now the proof is complete. □

Remark 3.1. The proof of Claim 2 is independent of the possibility of the existence of a queer component of the interior of M_f.

By Claims 1 and 2, we have the following $L^1(\omega)$ estimate

$$\begin{aligned}(3.3)\quad &\int_{\mathbb{P}^1}\left|\log|F_n| - d^{n-1}\cdot g_{I_{c_0}}\right|\omega\\ &\le \left(\omega(F_n^{-1}(\mathbb{D}(e^{-t_n})))t_n + C_0\right) + \omega(\mathbb{C}\setminus F_n^{-1}(\mathbb{D}(e^{-t_n})))t_n = t_n + C_0,\end{aligned}$$

so (1.2) holds.

Recalling (2.3), we also have $(d-1)F_n^*\delta_0 - d^n\cdot T_f = (d-1)\cdot \mathrm{dd}^c(\log|F_n| - d^{n-1}\cdot g_{I_{c_0}})$ on $\mathbb{P}^1$, so that by Green's theorem, for every $\phi\in C^2(\mathbb{P}^1)$, the estimate (3.3) yields

$$(1.3')\quad \left|\int_{\mathbb{P}^1}\phi\mathrm{d}\left((d-1)\cdot F_n^*\delta_0 - d^n\cdot T_f\right)\right| \le \left(\sup_{\mathbb{P}^1}\left|\frac{\mathrm{dd}^c\phi}{\omega}\right|\right)\cdot(d-1)(t_n + C_0),$$

so (1.3) holds. Now the proof of Theorem 1 is complete. □

4 - Proof of Theorem 2

Let $f:\mathbb{C}\times\mathbb{P}^1\to\mathbb{P}^1$ be the unicritical polynomials family of degree $d>1$ defined as (1.1). Recall the definitions (and properties) of $\Phi^*_{f,n}(\lambda,z)\in\mathbb{Z}[\lambda,z]$,

$p^*_{f,n}(\lambda,w)\in\mathbb{Z}[\lambda,z]$, and $\mathrm{Fix}^{**}_f(\lambda,n)$ in Subsection 1.2. For every $n\in\mathbb{N}$, it would be convenient to set

$$P^*_n(\lambda,w)=P^*_{f,n}(\lambda,w):=\frac{p^*_{f,n}(\lambda,w)}{d^{\nu(n)}}\in\mathbb{Q}[\lambda,w],$$

so that for every $w\in\mathbb{C}$, $P^*_n(\lambda,w)\in\mathbb{C}[\lambda]$ is *monic*.

Lemma 4.1. *For every $n\in\mathbb{N}$ and every $\lambda\in\mathbb{C}$, we have*

$$P^*_n(\lambda,0)=\big((-1)^{\nu(n)}\cdot\Phi^*_{f,n}(\lambda,0)\big)^{d-1}=\Big((-1)^{\nu(n)}\cdot\prod_{m\in\mathbb{N}:\,m|n}F_m(\lambda)^{\mu(n/m)}\Big)^{d-1}\tag{4.1}$$

(up to multiplication in n-th roots of unity). For every $n>1$, we have $0\notin(P^*_n(\cdot,0))^{-1}(0)$. For every $n\in\mathbb{N}$ and every $\lambda\in\mathbb{C}$, if $\lambda\in(P^*_n(\cdot,0))^{-1}(0)$, then $(c_0(\lambda)=)0\in\mathrm{Fix}^*_f(\lambda,n)$ and λ is a zero of $P^*_n(\cdot,0)$ of the order $d-1$.

Proof. For every $n\in\mathbb{N}$ and every $\lambda\in\mathbb{C}$, by the chain rule and the equalities $f'_\lambda(z)=d\cdot z^{d-1}$ and $\mathrm{Fix}^{**}_f(\lambda,n)=(\Phi^*_{f,n}(\lambda,\cdot))^{-1}(0)$, we have

$$(p^*_{f,n}(\lambda,0))^n\Big(=\prod_{z\in\mathrm{Fix}^{**}_f(\lambda,n)}(f^n_\lambda)'(z)\Big)=d^{\nu(n)n}\big((-1)^{\nu(n)}\cdot\Phi^*_{f,n}(\lambda,0)\big)^{n(d-1)}$$
$$=d^{\nu(n)n}\Big((-1)^{\nu(n)}\cdot\prod_{m\in\mathbb{N}:\,m|n}(f^m_\lambda(0)-0)^{\mu(n/m)}\Big)^{n(d-1)},$$

which (with the definition of F_m) yields (4.1). For every $m\in\mathbb{N}$, even by a direct computation, 0 is a simple zero of F_m in $\mathbb{C}$, so that for every $n>1$, $0\notin(P^*_n(\cdot,0))^{-1}(0)$ by $\sum_{m\in\mathbb{N}:\,m|n}\mu(n/m)=0$ and the latter equality in (4.1). For every $n\in\mathbb{N}$ and every $\lambda_0\in(P^*_n(\cdot,0))^{-1}(0)$, by the former equality in (4.1), we have $(c_0(\lambda_0)=)0\in\mathrm{Fix}^{**}_f(\lambda_0,n)$, which with $(f^n_\lambda)'(0)=(f^n_\lambda)'(c_0(\lambda))=0\neq1$ implies even $0\in\mathrm{Fix}^*_f(\lambda_0,n)$. Then by the latter equality in (4.1), λ_0 is a zero of $P^*_n(\cdot,0)$ of order $d-1$ since any zero of F_n is in fact simple. □

Recall the definitions of the sequences $(\sigma_0(n))$ and $(\sigma_1(n))$ in $\mathbb{N}$ (in Notation 1.2).

4.1 - *Proof of* (1.6)

For every $n \in \mathbb{N}$, the estimate (3.3) together with (1.4) and (4.1) yields the following $L^1(\omega)$ estimate

$$\int_{\mathbb{P}^1} \left|\log|P_n^*(\cdot,0)| - (d-1)\nu(n)\frac{g_{I_{c_0}}}{d}\right|\omega \leq t_n^* + (d-1)C_0 \cdot \sigma_0(n), \tag{4.2}$$

where we set

$$t_n^* := (d-1)\sum_{m\in\mathbb{N}:\, m|n} t_m = (2\log d)\sigma_1(n) + \Big((d+1)\log 2 - 2\log d + \frac{4C_{B_f}}{d-1} + (d-1)\log\big(\sqrt{2}+1\big)\Big)\sigma_0(n).$$

Recall that H_1 is by definition the component of H_f containing 0, and set

$$C_0^* := \pi + \int_0^\infty \frac{2r}{(1+r^2)^2}\,\log^+ \frac{r}{\sup\{t>0 : \mathbb{D}(t) \subset H_1\}}\,\mathrm{d}r = C_0 - \int_{H_1} G_{H_1}(\cdot,0)\omega.$$

In the rest of this subsection, for every $n > 1$, we also point out a slightly better estimate

$$\int_{\mathbb{P}^1} \left|\log|P_n^*(\cdot,0)| - (d-1)\nu(n)\frac{g_{I_{c_0}}}{d}\right|\omega \leq t_n^* + (d-1)C_0^* \tag{4.3}$$

than (4.2). In particular, by Green's theorem, for every $\phi \in C^2(\mathbb{P}^1)$ and every $n > 1$, we have

$$\left|\int_{\mathbb{P}^1} \phi\mathrm{d}\left(\mathrm{Per}_f^*(n,0) - \nu(n)\cdot T_f\right)\right| \leq \left(\sup_{\mathbb{P}^1}\left|\frac{\mathrm{dd}^c\phi}{\omega}\right|\right)\cdot\big(t_n^* + (d-1)C_0^*\big), \tag{1.6$'$}$$

which implies (1.6).

P r o o f. [Proof of (4.3)] For every $n \in \mathbb{N}$, by (4.1) and (3.1), we have

$$\sup_{B_f}\big|\log|P_n^*(\cdot,0)|\big| \leq t_n^*, \tag{3.1$'$}$$

which is a counterpart to (3.1). Fix $n > 1$. By (3.1′), $\inf_{\lambda\in B_f} |P_n^*(\lambda,0)| \geq e^{-t_n^*}$. As in the proof of Claim 1 in Section 3, let $\mathcal{F}^*$ be the family of all components of $(P_n^*(\cdot,0))^{-1}(\mathbb{D}(e^{-t_n^*}))$. By Lemma 4.1 and the description of H_f in Subsection 2.1, every $V \in \mathcal{F}^*$ is a piecewise real analytic Jordan domain in $H_f \setminus (I_{c_0} \cup H_1)$ now, and for every $V \in \mathcal{F}^*$, the restriction $P_n^*(\cdot,0)|V : V \to \mathbb{D}(t_n^*)$ is a proper holomorphic mapping of degree $d-1$ now and $\#(((P_n^*(\cdot,0))^{-1}(0)) \cap V) = 1$. For every $V \in \mathcal{F}^*$, letting λ_V be the unique point in $((P_n^*(\cdot,0))^{-1}(0)) \cap V$, by Myrberg's theorem [**22**], we now have

$$\log \frac{e^{-t_n^*}}{|P_n^*(\cdot,0)|} = G_{\mathbb{D}(e^{-t_n^*})}(P_n^*(\cdot,0),0) = (d-1)\cdot G_V(\cdot,\lambda_V) \quad \text{on } V.$$

Recalling $t_n^* \geq 0$, by a computation similar to that in the proof of Claim 1 in Section 3, we have

$$\int\limits_{(P_n^*(\cdot,w))^{-1}(\mathbb{D}(e^{-t_n^*}))} \left|\log|P_n^*(\cdot,0)| - \nu(n)(d-1)\frac{g_{I_{c_0}}}{d}\right|\omega \\ \leq \omega((P_n^*(\cdot,0))^{-1}(\mathbb{D}(e^{-t_n^*})))t_n^* + (d-1)C_0^*.$$

Moreover, by the same argument as that in the proof of Claim 2 in Section 3, we also have $\sup_{\mathbb{C}\setminus(P_n^*(\cdot,0))^{-1}(\mathbb{D}(e^{-t_n^*}))}\left|\log|P_n^*(\cdot,0)| - \nu(n)(d-1)d^{-1}g_{I_{c_0}}\right| \leq t_n^*$. Hence (4.3) holds. □

4.2 - *Proof of* (1.7)

As an application of (4.3), we also point out the following $L^1(\omega)$ estimate

$$(4.3') \qquad \int\limits_{\mathbb{P}^1}\left|\int\limits_0^{2\pi} \log|P_n^*(\lambda,re^{i\theta})|\frac{\mathrm{d}\theta}{2\pi} - \nu(n)(d-1)\frac{g_{I_{c_0}}}{d}\right|\omega(\lambda) \leq t_n^* + 2(d-1)C_0^*$$

for every $n > 1$ and every $r \in (0,1]$ (cf. [**3**, 2. in Theorem 3.1]). In particular, by Green's theorem, for every $\phi \in C^2(\mathbb{P}^1)$, every $n > 1$, and every $r \in (0,1]$, we will have

$$(1.7') \qquad \left|\int\limits_{\mathbb{P}^1} \phi \mathrm{d}\left(\int\limits_0^{2\pi} \mathrm{Per}_f^*(n,re^{i\theta})\frac{\mathrm{d}\theta}{2\pi} - \nu(n)\cdot T_f\right)\right| \\ \leq \left(\sup_{\mathbb{P}^1}\left|\frac{\mathrm{dd}^c\phi}{\omega}\right|\right)\cdot(t_n^* + 2(d-1)C_0^*),$$

which implies (1.7).

P r o o f. [Proof of (4.3′)] For every $n\in\mathbb{N}$ and every $\lambda\in\mathbb{C}\setminus(H_f\setminus I_{c_0})$, we have $\inf_{z\in\mathrm{Fix}_f^{**}(\lambda,n)}|(f_\lambda^n)'(z)|\ge 1$. Recall the description of components of $H_f\setminus I_{c_0}$ in Subsection 2.1. For every $n\in\mathbb{N}$, letting H_n^* be the union of all components U of $H_f\setminus I_{c_0}$ such that $n_U=n$ (so e.g. $H_1^*=H_1$), there is a holomorphic function $\lambda\mapsto z_\lambda$ on H_n^* such that for every $\lambda\in H_n^*$, $z_\lambda\in\mathrm{Fix}_f^{**}(\lambda,n)$ and that $(f_\lambda^n)'(z_\lambda)\equiv\phi_U(\lambda)$ on each component U of H_n^*. Fix $n>1$ and $r\in(0,1]$, and set $H_n^*(r):=\{\lambda\in H_n^*:(f_\lambda^n)'(z_\lambda)\in\mathbb{D}(r)\}=\bigcup_{U:\text{ a component of }H_n^*}\phi_U^{-1}(\mathbb{D}(r))$. For every $\lambda\in\mathbb{C}$, by the definitions of $P_{f,n}^*$ and $p_{f,n}^*$, we have

$$\int_0^{2\pi}\log|P_n^*(\lambda,re^{i\theta})|\frac{\mathrm{d}\theta}{2\pi}=\frac{1}{n}\sum_{z\in\mathrm{Fix}_f^{**}(\lambda,n)}\log\max\{r,|(f_\lambda^n)'(z)|\}-\nu(n)\log d$$
$$=\log|P_n^*(\lambda,0)|+\begin{cases}\displaystyle\frac{1}{n}\sum_{j=0}^{n-1}\log\frac{r}{|(f_\lambda^n)'(f_\lambda^j(z_\lambda))|} & \text{if }\lambda\in H_n^*(r),\\ 0 & \text{if }\lambda\in\mathbb{C}\setminus H_n^*(r),\end{cases}$$

which with (4.3) and the chain rule yields

$$\int_{\mathbb{P}^1}\left|\int_0^{2\pi}\log|P_n^*(\lambda,re^{i\theta})|\frac{\mathrm{d}\theta}{2\pi}-\nu(n)(d-1)\frac{g_{I_{c_0}}}{d}\right|\omega(\lambda)$$
$$\le\left(t_n^*+(d-1)C_0^*\right)+\int_{H_n^*(r)}\log\frac{r}{|(f_\lambda^n)'(z_\lambda)|}\,\omega(\lambda).$$

For every component V of $H_n^*(r)$, letting U be the component of $H_n^*(=H_n^*(1))$ containing V, the restriction $\phi_U|V:V\to\mathbb{D}(r)$ is a proper holomorphic mapping of degree $d-1$, so letting λ_V be the unique point in $V\cap\phi_U^{-1}(0)$, by Myrberg's theorem [**22**], we have

$$\log\frac{r}{|(f_\lambda^n)'(z_\lambda)|}=G_{\mathbb{D}(r)}((\phi_U|V)(\lambda),0)=(d-1)\cdot G_V(\lambda,\lambda_V)\quad\text{on }V.$$

Noting that $H_n^*\subset H_f\setminus(I_{c_0}\cup H_1)$, by a computation similar to that in the proof of Claim 1 in Section 3, we have

$$\int_{H_n^*(r)}\log\frac{r}{|(f_\lambda^n)'(z_\lambda)|}\,\omega(\lambda)\le(d-1)\cdot C_0^*.$$

Hence (4.3′) holds. □

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Yûsuke Okuyama
Division of Mathematics
Kyoto Institute of Technology
Sakyo-ku, Kyoto 606-8585 Japan
e-mail: okuyama@kit.ac.jp