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Mixed norm Lebesgue spaces with variable exponents and applications

Abstract. In this paper, we introduce the mixed norm Lebesgue spaces with variable exponent. We use this family of function spaces to study Calderón-Zygmund operators on product domains, the Littlewood-Paley operators associated with family of disjoint rectangles, the non-tangential maximal function and the area function for bi-harmonic functions, the Ricci-Stein singular integrals and the characterizations of the function space of bounded mean oscillation.

Keywords. Mixed-norm spaces, Lebesgue spaces with variable exponent, extrapolation, product domains, Calderón-Zygmund operators, Littlewood-Paley operators, bi-harmonic functions, Ricci-Stein singular integrals, BMO.

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Contents

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1 - Introduction

This paper aims to study the mixed norm Lebesgue spaces with variable exponent. The study of this family of function spaces is motivated by the results in [**21**, **29**]. In [**29**], we find that the strong maximal operator is bounded on the Lebesgue space with variable exponent $L^{p(\cdot)}$ if and only if $p(\cdot) = q$ with $1 < q \leq \infty$. On the other hand, in [**21**, Theorem 4.3], we obtain the boundedness of the strong maximal operator on the mixed-norm Lebesgue spaces with variable exponent $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ (see Definition 3.1).

Therefore, the results from [**21**] give us inspiration to study those operators and estimates related to the strong maximal operator on $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$. That is, those operators and estimates associated with the multiparameter dilation

$$(x_i)_{i=1}^n \to (\delta_i x_i)_{i=1}^n, \quad \delta_i > 0,\ 1 \leq i \leq n. \tag{1}$$

More precisely, we investigate the boundedness of the Calderón-Zygmund operators on product domains, the Littlewood-Paley operators associated with family of disjoint rectangles on $\mathbb{R}^2$ and the estimates of the non-tangential maximal function and the area function for bi-harmonic functions.

Our approach does not only apply to the estimates for multiparameter dilations (1), it also applies to the estimates related to the Zygmund dilation $(x_1, x_2, x_3) \to (\delta_1 x_1, \delta_2 x_2, \delta_1\delta_2 x_3)$, $\delta_1, \delta_2 > 0$. We demonstrate the application on the Zygmund dilation by studying the Ricci-Stein singular integrals on $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

In addition, our approach also applies to the one parameter dilation

$$(x_i)_{i=1}^n \to (\delta x_i)_{i=1}^n, \quad \delta > 0.$$

We present this application via the characterizations of the function space of bounded mean oscillation BMO in terms of $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

We obtain the above results by using extrapolation theory. The extrapolation theory is a powerful tool in harmonic analysis. One of the special features of extrapolation theory is its application on nonlinear operators and norm inequalities that do not involve operators. In this paper, that kinds of applications are appeared on the John-Nirenberg inequalities for $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ and characterizations of BMO on terms of $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

Our approach applies to the general multiparameter dilation (1). On the other hand, to simplify our presentation, we just consider the case $(x_1, x_2) \to (\delta_1 x_1, \delta_2 x_2)$, $\delta_1, \delta_2 > 0$, except for the study of the Ricci-Stein singular integrals.

This paper is organized as follows. In section 2, we present some preliminaries and definitions for our study such as the definitions of Muckenhoupt weight function on product domains and Lebesgue spaces with variable exponent. We introduce the mixed-norm Lebesgue spaces with variable exponent in Section 3. The extrapolation theory for the mixed-norm Lebesgue spaces with variable exponent is also presented in this section.

In Section 4, we establish the boundedness of the Calderón-Zygmund operators on product domains, the Littlewood-Paley operators associated with family of disjoint rectangles on $\mathbb{R}^2$, the estimates of the non-tangential maximal function and the area function for bi-harmonic functions, the boundedness of the Ricci-Stein singular integrals on $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ and the characterizations of the function space of bounded mean oscillation BMO in terms of $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

2 - Preliminaries and Definitions

For any $x \in \mathbb{R}$ and $r > 0$, let $B(x,r) = \{y \in \mathbb{R} : |x-y| < r\}$ and $\mathbb{B} = \{B(x,r) : x \in \mathbb{R},\ r > 0\}$. For any $B \in \mathbb{B}$, denote the center and the radius of B by x_B and r_B, respectively.

For any Lebesgue measurable set E, we write χ_E and $|E|$ for the characteristic function of E and the Lebesgue measure of E, respectively.

Let L^1_{loc} denote the family of locally Lebesgue integrable functions. Let L^∞_{comp} and C^∞_c be the sets of bounded functions with compact support and smooth functions with compact support, respectively. Let $\mathcal{M}$ denote the set of Lebesgue measurable functions.

For any $r, s > 0$ and $z = (x,y) \in \mathbb{R}\times\mathbb{R}$, define $R(z,r,s) = B(x,r)\times B(y,s)$. Write $\mathcal{R} = \{R(z,r,s) : z \in \mathbb{R}\times\mathbb{R}, s, r > 0\}$.

The strong maximal operator M_S is defined as

$$\mathrm{M}_S\, f(z) = \sup_{R \ni z} \frac{1}{|R|} \int_R |f(u)| du, \quad f \in L^1_{loc}$$

where the supremum is taking over all $R \in \mathcal{R}$ containing z.

The reader is referred to [**38**, Chapter II, Sections 5.20-5.23] for some important results about the strong maximal operator on Lebesgue spaces.

Definition 2.1. For $1 < p < \infty$, a locally integrable function $\omega : \mathbb{R} \to [0,\infty)$ is said to be an A_p weight if

$$[\omega]_{A_p} = \sup_{B\in\mathbb{B}} \left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty$$

where $p' = \frac{p}{p-1}$. A locally integrable function $\omega : \mathbb{R} \to [0,\infty)$ is said to be an

A_1 weight if

$$[\omega]_{A_1} = \sup_{B\in\mathbb{B}} \Big(\frac{1}{|B|}\int_B \omega(y)dy\Big) \operatorname*{ess\,sup}_{x\in B} \omega(x)^{-1} < \infty \tag{2}$$

for some constant $C > 0$. We define $A_\infty = \cup_{p\geq 1} A_p$.

We now recall the corresponding weight functions for the product domain $\mathbb{R}\times\mathbb{R}$ from [**12**, Chapter IV, Section 6].

Definition 2.2. For $1 < p < \infty$, we say that a nonnegative locally integrable function $\omega \in A_p^*$ if

$$[\omega]_{A_p^*} = \sup_{R\in\mathcal{R}} \left(\frac{1}{|R|}\int_R \omega(z)dz\right)\left(\frac{1}{|R|}\int_R \omega(z)^{-\frac{1}{p-1}}dz\right)^{p-1} < \infty.$$

We say that a nonnegative measurable function $\omega \in A_1^*$ if

$$[\omega]_{A_1^*} = \sup_{R\in\mathcal{R}} \left(\frac{1}{|R|}\int_R \omega(z)dz\right) \operatorname*{ess\,sup}_{z\in R} \omega(z)^{-1} < \infty.$$

We write $A_\infty^* = \cup_{1\leq p<\infty} A_p^*$.

Let $u : \mathbb{R} \to [0,\infty)$ and $v : \mathbb{R} \to [0,\infty)$. It follows from the definition of A_p^* that

$$u, v \in A_p \Rightarrow uv \in A_p^*. \tag{3}$$

We recall the definition of the Lebesgue space with variable exponents from [**5**, **7**]. For any Lebesgue measurable function $p : \mathbb{R} \to [1,\infty)$, the Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of all $f \in \mathcal{M}$ such that

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : \rho_p(f/\lambda) \leq 1\} < \infty$$

where

$$\rho_p(f) = \int_{\mathbb{R}} |f(x)|^{p(x)}dx.$$

We call $p(x)$ the exponent function of $L^{p(\cdot)}$. The reader is referred to [**5**, **7**] for some basic properties of $L^{p(\cdot)}$. Particularly, $L^{p(\cdot)}$ is a Banach function space, see [**7**, Theorem 3.2.13].

The associated space of $L^{p(\cdot)}$ is given by $L^{p'(\cdot)}$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ [**7**, Theorem 3.2.13]. The reader is referred to [**7**, Definition 2.7.1] for the definition of associate space.

Write

$$p_- = \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}\} \quad \text{and} \quad p_+ = \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}\}.$$

Let $\mathcal{P}$ denote the set of Lebesgue measurable functions $p : \mathbb{R} \to [1, \infty)$ satisfying $1 < p_- \leq p_+ < \infty$.

Definition 2.3. Let M denote the Hardy-Littlewood maximal operator. For any exponent function $p(\cdot) : \mathbb{R} \to [1, \infty)$, we write $p(\cdot) \in \mathcal{B}$ if M is bounded on $L^{p(\cdot)}$.

We recall the following characterization of $\mathcal{B}$ given by Diening in [**6**].

Theorem 2.1. *Let $p(\cdot) \in \mathcal{P}$. Then the following conditions are equivalent:*

1. $p(\cdot) \in \mathcal{B}$.
2. $p'(\cdot) \in \mathcal{B}$.
3. *there exists a* $1 < q_0 < p_-$ *such that* $p(\cdot)/q \in \mathcal{B}$ *for all* $1 < q < q_0$.
4. *there exists a* $1 < q_0 < p_-$ *such that* $(p(\cdot)/q)' \in \mathcal{B}$ *for all* $1 < q < q_0$.

Proof. The equivalence of Items (1) and (2) and the implications, (3),(4)$\Rightarrow$ (1),(2), follow from [**6**]. It remains to show the implications (1),(2)$\Rightarrow$ (3),(4). In view of [**6**], Items (1) and (2) are equivalent to

$$p(\cdot)/q \in \mathcal{B} \quad \text{for} \quad \text{some} \quad 1 < q < p_-, \tag{4}$$

$$(p(\cdot)/q)' \in \mathcal{B} \quad \text{for} \quad \text{some} \quad 1 < q < p_-. \tag{5}$$

In view of Jensen's inequality, for any $1 < r < \infty$, we have

$$(\mathrm{M} f)^r \leq \mathrm{M}(|f|^r).$$

Whenever $p(\cdot)/q \in \mathcal{B}$, for any $r < q$, by using of Jensen's inequality, we have

$$(\mathrm{M} f)^{q/r} \leq \mathrm{M}(|f|^{q/r}).$$

Consequently,

$$\begin{aligned}\|\mathrm{M} f\|_{L^{p(\cdot)/r}} &= \|(\mathrm{M} f)^{q/r}\|_{L^{p(\cdot)}/q} \leq \|\mathrm{M}(|f|^{q/r})\|_{L^{p(\cdot)}/q} \\ &\leq C\||f|^{q/r}\|_{L^{p(\cdot)}/q} = C\|f\|_{L^{p(\cdot)}/r}.\end{aligned}$$

Therefore, Item (3) of Theorem 2.1 is valid. Then, Item (4) of Theorem 2.1 follows from the equivalence of Items (1) and (2) of Theorem 2.1. □

3 - Mixed norm Lebesgue spaces with variable exponents

We introduce the mixed-norm Lebesgue spaces with variable exponent in this section. We also study the duality, the density of continuous functions with compact support and the Fatou's lemma in the following. Most importantly, we also present the extrapolation theory for the mixed-norm Lebesgue spaces with variable exponent in this section.

We now introduce the mixed-norm Lebesgue spaces with variable exponent.

Definition 3.1. Let $p_1(\cdot), p_2(\cdot) : \mathbb{R} \to [1, \infty)$ be Lebesgue measurable functions. The mixed-norm Lebesgue space with variable exponent $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ consists of all Lebesgue measurable function on $\mathbb{R} \times \mathbb{R}$, f satisfying

$$\|f\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})} = \| \|f\|_{L^{p_1(\cdot)}} \|_{L^{p_2(\cdot)}} < \infty.$$

The reader is referred to [**1**, **2**, **4**, **10**, **11**, **14**, **21**, **33**, **39**] for the studies of mixed norm spaces in different directions of researches such as the embedding of Sobolev spaces, inclusion problem and the interpolation theory.

Since $L^{p_2(\cdot)}$ possesses the Fatou property [**7**, Theorem 2.3.17 and p.77], in view of Luxemburg-Gribanov theorem [**31**], $\|f\|_{L^{p_2(\cdot)}}$ is Lebesgue measurable, therefore, $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ is well defined. Furthermore, $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ is a Banach space, see [**4**, p.158].

Whenever $0 \leq f_j \uparrow f$, we have $\|f_j\|_{L^{p_2(\cdot)}} \uparrow \|f\|_{L^{p_2(\cdot)}}$. Consequently, the Fatou property of $L^{p_1(\cdot)}$ guarantees that

$$\|f_j\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})} \uparrow \|f\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})}. \tag{6}$$

We now define the associate space of $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

Definition 3.2. Let $L^{p_1(\cdot)}$ and $L^{p_2(\cdot)}$ be Lebesgue spaces with variable exponent on $\mathbb{R}$. The associated space $(L^{p_1(\cdot)}, L^{p_2(\cdot)})'$ consists of all $f \in \mathcal{M}$ such that

$$\|f\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})'} = \sup \left\{ \left| \int_{\mathbb{R}\times\mathbb{R}} f(x)g(x)\, dx \right| : \|g\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})} \leq 1 \right\} < \infty.$$

The following result identifies the associate space of $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

Proposition 3.1. *Let $L^{p_1(\cdot)}$ and $L^{p_2(\cdot)}$ be Lebesgue spaces with variable exponent on $\mathbb{R}$. The associate space of $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ is $(L^{p_1'(\cdot)}, L^{p_2'(\cdot)})$.*

The above result is a special case of a general result for Banach function space. Therefore, for brevity, we refer the reader to [**4**, **14**] for details.

In view of Definition 3.2 and Lemma 3.1, we have the Hölder inequality for $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

There exists a constant $C > 0$ such that for any $f \in (L^{p_1(\cdot)}, L^{p_2(\cdot)})$ and $g \in (L^{p_1'(\cdot)}, L^{p_2'(\cdot)})$, we have

$$\int |f(x)g(x)dx \leq C\|f\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}\|g\|_{(L^{p_1'(\cdot)},L^{p_2'(\cdot)})}. \tag{7}$$

For any pair of exponent functions (p_1, p_2), write

$$(p_1,p_2)_+ = \max((p_1)_+,(p_2)_+) \quad \text{and} \quad (p_1,p_2)_- = \min((p_1)_-,(p_2)_-).$$

Lemma 3.1. *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$. For any $k, l \in \mathbb{N}$, there exist constants $C_{k,l}, D_{k,l} > 0$ such that*

$$\begin{aligned} C_{k,l}\|f\chi_{R(0,k,l)}\|_{L^{(p_1,p_2)_-}} &\leq \|f\chi_{R(0,k,l)}\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \\ &\leq D_{k,l}\|f\chi_{R(0,k,l)}\|_{L^{(p_1,p_2)_+}}. \end{aligned} \tag{8}$$

Proof. In view of [**5**, Corollary 2.50], we have constants c_k, d_k such that for any $y \in B(0,l) \subset \mathbb{R}$, we have

$$c_k\|f\chi_{B(0,k)}(\cdot,y)\|_{L^{(p_1)_-}} \leq \|f\chi_{B(0,k)}(\cdot,y)\|_{L^{p_1(\cdot)}} \leq d_k\|f\chi_{B(0,k)}(\cdot,y)\|_{L^{(p_1)_+}}.$$

Applying [**5**, Corollary 2.50] to $L^{p_2(\cdot)}$, we obtain

$$C_{k,l}\|f\chi_{R(0,k,l)}\|_{L^{(p_1,p_2)_-}} \leq \|f\chi_{R(0,k,l)}\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \leq D_{k,l}\|f\chi_{R(0,k,l)}\|_{L^{(p_1,p_2)_+}}$$

for some constants $C_{k,l}, D_{k,l} > 0$. ☐

Even though $L^{p_1(\cdot)}$ and $L^{p_2(\cdot)}$ are Banach function spaces, the set of simple functions is not necessary a subset of $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$. This is true even $p(\cdot)$ and $q(\cdot)$ are constant functions.

Proposition 3.2. *Let $1 < p, q < \infty$.*

1. *If $q < p$, then there exists a Lebesgue measurable set E such that $\chi_E \notin (L^p, L^q)$.*

2. *If $q > p$, then there exists a Lebesgue measurable set E such that $\chi_E \notin (L^p, L^q)'$. That is, there exists a Lebesgue measurable set E such that*

$$\sup_{\|g\|\leq 1, g\in(L^p,L^q)} \int_E |g(x,y)|dxdy = \infty.$$

Proof. We first consider the case $q < p$. Define

$$E = \bigcup_{k=1}^{\infty} \left[k, k + \frac{1}{k^{p/q}}\right] \times [k, k+1].$$

As $q < p$, we have

$$|E| = \sum_{k=1}^{\infty} \frac{1}{k^{p/q}} < \infty.$$

On the other hand, we find that

$$\begin{aligned}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_E(x,y)dx \right)^{q/p} dy &= \int_{\mathbb{R}} \left(\sum_{k=1}^{\infty} \frac{1}{k^{p/q}} \chi_{[k,k+1]}(y) \right)^{q/p} dy \\ &= \int_{\mathbb{R}} \sum_{k=1}^{\infty} \frac{1}{k} \chi_{[k,k+1]}(y)dy = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.\end{aligned}$$

Therefore, $\chi_E \notin (L^p, L^q)$.

When $q > p$, we have $q' < p'$. The above result shows that there exists a Lebesgue measurable set E such that $\chi_E \notin (L^{p'}, L^{q'})$. Since $(L^p, L^q)' = (L^{p'}, L^{q'})$, we have $\chi_E \notin (L^p, L^q)'$. □

The above result shows that $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ is not necessary a Banach function space in the sense of [**3**, Definitions 1.1 and 1.3]. Especially, $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ does not necessary satisfy [**3**, Definition 1.1 (P4) and (P5)].

On the other hand, it is obvious that $\chi_E \in (L^{p_1(\cdot)}, L^{p_2(\cdot)})$ when E is a bounded Lebesgue measurable set. Therefore, $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ is a ball Banach function space, see [**37**].

Next, we establish the Fatou's lemma for $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

Lemma 3.2. *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$. If $\{f_k\} \subset (L^{p_1(\cdot)}, L^{p_2(\cdot)})$ satisfies*

$$\lim_{k\to\infty} f_k = f \quad \text{a.e.} \quad \text{and} \quad \liminf_{k\to\infty} \|f_k\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})} < \infty.$$

Then, $f \in (L^{p_1(\cdot)}, L^{p_2(\cdot)})$ and

$$\|f\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})} \le \liminf_{k\to\infty} \|f_k\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})}.$$

Proof. Define

$$g_k = \inf_{m \ge k} |f_m|, \quad \forall k \in \mathbb{N}.$$

We have $|g_k| \leq |f_m|$, for all $m \geq k$, $g_k \in (L^{p_1(\cdot)}, L^{p_2(\cdot)})$ and $g_k \leq g_{k+1}$, for all $k \in \mathbb{N}$. Moreover, $\lim_{k\to\infty} g_k = \liminf_{m\to\infty} |f_m| = |f|$ a.e.

According to the Fatou property for $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ (6), we obtain

$$\begin{aligned}\|f\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} &= \lim_{k\to\infty} \|g_k\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \leq \lim_{k\to\infty} \inf_{m\geq k} \|f_m\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \\ &= \lim \inf_{k\to\infty} \|f_k\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} < \infty.\end{aligned}$$

Therefore, $f \in (L^{p_1(\cdot)}, L^{p_2(\cdot)})$. □

For any open set $\Omega \subseteq \mathbb{R}^2$, let $C_c(\Omega)$ the set of continuous functions with compact support in Ω. We show that $C_c(\mathbb{R}^2)$ is dense in $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

Lemma 3.3. *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{P}$. The set $C_c(\mathbb{R}^2)$ is dense in $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.*

Proof. For any $f \in (L^{p_1(\cdot)}, L^{p_2(\cdot)})$, we have $f = f_+ - f_-$ where $f_+ = \max(f, 0)$ and $f_- = -\min(f, 0)$ and $f_+, f_- \in (L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

Let $T_k = R(0, k, k) \cap \{(x, y) \in \mathbb{R}^2 : 0 < f_+(x, y) \leq k\}$, $k \in \mathbb{N}$. Define $f_k = f_+\chi_{T_k} \uparrow f_+$. We have $0 \leq f_+ - f_k \leq f_+$, $k \in \mathbb{N}$ and $|f_+ - f_k| \downarrow 0$ a.e. As $\|\cdot\|_{L^{p_1(\cdot)}}$ is an absolutely continuous norm [**3**, Chapter 1, Definition 3.1 and Proposition 3.5] [**5**, p.73], we find that $\|f_+ - f_k\|_{L^{p_1(\cdot)}} \downarrow 0$ a.e. and $\|f_+ - f_k\|_{L^{p_1(\cdot)}} \leq \|f_+\|_{L^{p_1(\cdot)}} \leq \|f\|_{L^{p_1(\cdot)}}$ a.e. In addition, as $\|\cdot\|_{L^{p_2(\cdot)}}$ is also absolutely continuous and $\|f\|_{L^{p_1(\cdot)}} \in L^{p_2(\cdot)}$, we have

$$\|f_+ - f_k\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} = \|\|f_+ - f_k\|_{L^{p_1(\cdot)}}\|_{L^{p_2(\cdot)}} \downarrow 0.$$

That is, for any $\epsilon > 0$, there is a $k \in \mathbb{N}$ such that $\|f_+ - f_k\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} < \epsilon/2$. Moreover, f_k is a bounded function with compact support. Therefore, $f_k \in L^{(p_1,p_2)_+}(R(0, k, k))$.

Since $p_1^+, p_2^+ < \infty$, we have $(p_1, p_2)_+ = \max((p_1)_+, (p_2)_+) < \infty$. Furthermore, $C_c(R(0, k, k))$ is dense in $L^{(p_1,p_2)_+}(R(0, k, k))$. Therefore, there exists a $g \in C_c(\mathbb{R}^2)$ such that

$$\|g - f_k\|_{L^{(p_1,p_2)_+}(R(0,k,k))} \leq \frac{\epsilon}{2(1 + D_{k,k})}$$

where $D_{k,k}$ is the constant given by (8).

Lemma 3.1 assures that

$$\begin{aligned}\|f_+ - g\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} &\leq \|f_+ - f_k\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} + \|f_k - g\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \\ &\leq \frac{\epsilon}{2} + D_{k,k}\|f_k - g\|_{L^{(p_1,p_2)_+}(R(0,k,k))} < \epsilon.\end{aligned}$$

Similarly, we have a $h \in C_c(\mathbb{R}^2)$ such that $\|f_- - h\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} < \epsilon$. Consequently,

$$\|f - (g - h)\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \le \|f_+ - g\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} + \|f_- - h\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} < 2\epsilon$$

and $g - h \in C_c(\mathbb{R}^2)$. Therefore, $C_c(\mathbb{R}^2)$ is dense in $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$. □

As shown in [**29**], the strong maximal operator is bounded on $L^{p(\cdot)}$ if and only if $p(\cdot) = p$ with $1 < p \le \infty$. On the other hand, according to [**21**, Theorem 4.3], we have the boundedness of the strong maximal operator on $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

Theorem 3.1. *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{B}$. There exists a constant $C > 0$ such that for any $f \in (L^{p_1(\cdot)}, L^{p_2(\cdot)})$, we have*

$$\| \mathrm{M}_S f\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \le C\|f\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}.$$

The boundedness of the strong maximal operator on $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ yields the following estimate.

Lemma 3.4. *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{B}$. There exists a constant $C > 0$ such that for any $R \in \mathcal{R}$, we have*

$$|R| \le \|\chi_R\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}\|\chi_R\|_{(L^{p'_1(\cdot)},L^{p'_2(\cdot)})} \le C|R|. \tag{9}$$

Proof. The Hölder inequality (7) yields the first inequality in (9).

For any $R \in \mathcal{R}$, we consider the projection

$$(P_R g)(y) = \left(\frac{1}{|R|}\int_R |g(x)|dx\right)\chi_R(y).$$

There exists a constant $C > 0$ such that for any $R \in \mathcal{R}$, $P_R(f) \le C\,\mathrm{M}_S(f)$. Hence,

$$\sup_R \|P_R\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})\to(L^{p_1(\cdot)},L^{p_2(\cdot)})} \le C\| \mathrm{M}_S \|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})\to(L^{p_1(\cdot)},L^{p_2(\cdot)})}.$$

The definition of associate space assures that

$$\begin{aligned}
&\|\chi_R\|_{(L^{p'_1(\cdot)},L^{p'_2(\cdot)})}\|\chi_R\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \\
&= \sup\left\{\Big|\int_R g(x)dx\Big|\|\chi_R\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} : g \in (L^{p_1(\cdot)}, L^{p_2(\cdot)}),\ \|g\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \le 1\right\} \\
&\le \sup\left\{|R|\|P_R g\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} : g \in (L^{p_1(\cdot)}, L^{p_2(\cdot)}),\ \|g\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \le 1\right\} \\
&\le |R|\sup\left\{\| \mathrm{M}_S g\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} : g \in (L^{p_1(\cdot)}, L^{p_2(\cdot)}),\ \|g\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \le 1\right\}.
\end{aligned}$$

In view of Theorem 3.1, we have

$$\begin{aligned}&\|\chi_R\|_{(L^{p_1'(\cdot)},L^{p_2'(\cdot)})}\|\chi_R\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}\\ &\le |R|\|\mathrm{M}_S\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})\to(L^{p_1(\cdot)},L^{p_2(\cdot)})}\le C|R|.\end{aligned}$$

□

The estimates (9) is an crucial component to establish the characterization of BMO via $(L^{p_1(\cdot)},L^{p_2(\cdot)})$ in the next section.

At the end of this section, we present the extrapolation theory for the mixed-norm Lebesgue spaces with variable exponent.

Theorem 3.2. *Let $p_1(\cdot),p_2(\cdot)\in\mathcal{P}$. Given a family $\mathcal{F}$, suppose that for some $0<p_0<\infty$ and for every $\omega_0\in A_1^*$, we have*

$$\int\limits_{\mathbb{R}\times\mathbb{R}} f(x,y)^{p_0}\omega_0(x,y)dxdy\le C\int\limits_{\mathbb{R}\times\mathbb{R}} g(x,y)^{p_0}\omega_0(x,y)dxdy \tag{10}$$

for any $(f,g)\in\mathcal{F}$ where C depends only on p_0 and $[\omega_0]_{A_1^*}$.

Suppose that there exists $p_0\le q_0<(p_1,p_2)_-$ such that

$$(p_1(\cdot)/q_0)',(p_2(\cdot)/q_0)'\in\mathcal{B},$$

then,

$$\|f\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}\le C\|g\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})},\quad (f,g)\in\mathcal{F}. \tag{11}$$

We refer the reader to [**21**, Theorem 3.2] for the proof of the above result.

4 - Applications

4.1 - *Calderón-Zygmund operators on product domains*

In this section, we study the Calderón-Zygmund operators associated with the multiparameter dilation (1). We first recall the definition of the Calderón-Zygmund operator on product domains from [**8**, **28**].

Let $\epsilon>0$. For any bounded linear operator T on $L^2(\mathbb{R})$, it is a Calderón-Zygmund operator if

1. there exists a kernel $k(x,y):\mathbb{R}\times\mathbb{R}\backslash\{(x,x):x\in\mathbb{R}\}\to\mathbb{R}$ such that for any $f,g\in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}f\cap\mathrm{supp}g=\emptyset$, we have

$$\int g(x)Tf(x)dx=\int\int g(x)k(x,y)f(y)dxdy.$$

2. There exists a constant $C > 0$ such that for any $\gamma > 2$, the kernel $k(x, y)$ satisfies

$$\left(\int_{|x-y|>\gamma|x-z|} |k(x,y) - k(z,y)|^2 dy \right)^{1/2} \leq C\gamma^{-\epsilon-1/2}|x-z|^{-1/2}. \tag{12}$$

If T is a Calderón-Zygmund operator, we write $T \in CZ^2_\epsilon(\mathbb{R})$ and define

$$\|T\|_{CZ} = \|T\|_{L^2 \to L^2} + \inf C$$

where the infimum on the right hand side is taken on those constant C given in (12).

For any bounded linear operator T on $L^2(\mathbb{R}^2)$, we write $T \in CZ^2_\epsilon(\mathbb{R} \times \mathbb{R})$ if it satisfies the following conditions.

1. For any fixed $(x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{R}$, there exist linear operators

$$k_1(x_1, y_1), k_2(x_2, y_2) \in CZ^2_\epsilon(\mathbb{R})$$

such that for any $f_i, g_i \in C_c^\infty(\mathbb{R})$, $i = 1, 2$, we have

$$\begin{aligned} &\int\int g_1(x_1)g_2(x_2)T(f_1f_2)(x_1,x_2)dx_1, dx_2 \\ &= \int\int g_1(x_1)\langle g_2 k_1(x_1,y_1)(f_2)\rangle f_1(y_1)dx_1dy_1 \end{aligned}$$

when $\mathrm{supp} g_1 \cap \mathrm{supp} f_1 = \emptyset$ and

$$\begin{aligned} &\int\int g_1(x_1)g_2(x_2)T(f_1f_2)(x_1,x_2)dx_1, dx_2 \\ &= \int\int g_2(x_2)\langle g_1 k_2(x_1,y_2)(f_1)\rangle f_2(y_2)dx_2dy_2 \end{aligned}$$

when $\mathrm{supp} g_2 \cap \mathrm{supp} f_2 = \emptyset$.

2. There exist constants $C_1, C_2 > 0$ such that for any $\gamma > 0$, the operators $k_1(x_1, y_1), k_2(x_2, y_2)$ satisfy

$$\begin{aligned} &\left(\int_{|x_i-y_i|>\gamma|x_i-z_i|} \|k_i(x_i,y_i) - k_i(z_i,y_i)\|^2_{CZ} dy_i \right)^{1/2} \\ &\leq C_i\gamma^{-\epsilon-1/2}|x_i - z_i|^{-1/2}, \qquad i = 1, 2. \end{aligned}$$

We now have the boundedness of the Calderón-Zygmund operators on the mixed-norm Lebesgue space with variable exponent $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

Theorem 4.1. *Let $\epsilon > 0$ and $p_1(\cdot), p_2(\cdot) \in \mathcal{B}$. If $T \in CZ^2_\epsilon(\mathbb{R}\times\mathbb{R})$, then T can be extended to be a bounded linear operator on $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.*

Proof. In view of the weighted norm inequalities for Calderón-Zygmund operators on product domain given by Fefferman in [**8**], for any $1 < p < \infty$ and $\omega \in A_1^* \subset A_p^*$, we have

$$\int |Tf(x,y)|^p \omega(x,y)dxdy \le C\int |f(x,y)|^p dxdy.$$

According to Theorem 2.1, there exists a $1 < q_0 < (p_1, p_2)_-$ such that $(p_1/q_0)', (p_2/q_0)' \in \mathcal{B}$. We apply Theorem 3.2 with $p_0 = q_0$ on the set

$$\mathcal{F} = \{(Tf, f) : f \in C_c(\mathbb{R}^2)\}.$$

Therefore, there exists a $C > 0$ such that for any $f \in C_c(\mathbb{R}^2)$,

$$\|Tf\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \le C\|f\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}. \tag{13}$$

In view of Lemma 3.3, $C_c(\mathbb{R}^2)$ is dense in $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$, therefore, (13) guarantees that T can be extended to be a bounded linear operator on $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$. $\square$

4.2 - *Rubio de Francia inequalities*

The one parameter Rubio de Francia inequalities study the mapping property of the Littlewood-Paley operator associated with intervals on $\mathbb{R}$, see [**35**]. The extension of the Rubio de Francia inequalities on product domains is given in [**30**]. In this section, we further generalize it to $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

Let $\mathcal{S}'(\mathbb{R}^2)$ denote the set of Schwartz distributions on $\mathbb{R}^2$. For any $f \in \mathcal{S}'(\mathbb{R}^2)$, we denote the Fourier transform of f by $\hat{f}$.

Let $W = \{R_j\}$ be a collection of disjoint rectangles in $\mathbb{R}\times\mathbb{R}$ with side parallel to the coordinate axes. The Littlewood-Paley operator associated with W is defined as

$$\Delta f(x) = \left(\sum_{R_j\in W} |S_{R_j}(f)(x)|^2\right)^{1/2}$$

where

$$(S_{R_j}f)^\wedge(\xi) = \chi_{R_j}(\xi)\hat{f}(\xi).$$

We have the following result from [**30**, Section 4].

Theorem 4.2. *Let $2 < p < \infty$. If $\omega \in A^*_{p/2}$, then*

$$\|\Delta f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.$$

We are ready to extend the Rubio de Francia inequalities to $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

Theorem 4.3. *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{B}$. If $(p_1, p_2)_- > 2$, then the Littlewood-Paley operator Δ can be extended to be a bounded operator on $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.*

Proof. As $(p_1, p_2)_- > 2$, in view of Theorem 2.1, there exists a $2 < q_0 < p_1, p_2)_-$ such that $(p_1(\cdot)/q_0)', (p_2/q_0)' \in \mathcal{B}$. Moreover, Theorem 4.2 assures that for any $\omega \in A^*_1 \subset A^*_{q_0/2}$, we have

$$\|\Delta f\|_{L^{q_0}(\omega)} \leq C\|f\|_{L^{q_0}(\omega)}.$$

Therefore, we are allowed to apply Theorem 3.2 with $p_0 = q_0$ to

$$\mathcal{F} = \{(\Delta f, f) : f \in C_c(\mathbb{R}^2)\}.$$

We obtain a constant $C > 0$ such that for any $f \in C_c(\mathbb{R}^2)$

$$\|\Delta f\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})} \leq C\|f\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})}.$$

Since $C_c(\mathbb{R}^2)$ is dense in $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ and Δ is a sublinear operator, the above inequality assures that Δ can be extended to be a bounded operator on $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$. $\square$

4.3 - *Biharmonic functions*

The study of the nontangential maximal function and the area function for harmonic functions is one of the main topics in harmonic analysis. This study had been extended to multiparameter setting in [**15**]. In this section, we further extend it to setting of variable exponent analysis.

Write $\mathbb{R}^2_+ = \mathbb{R} \times (0, \infty)$. For any $x = (x_1, x_2) \in \mathbb{R}^2$, define

$$\Gamma_i(x_i) = \{(y_i, t_i) \in \mathbb{R}^2_+ : |x_i - y_i| \leq t_i\}, \quad i = 1, 2$$

and

$$\Gamma(x) = \Gamma_1(x_1) \times \Gamma_2(x_2).$$

For any Lebesgue measurable function f on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$, the nontangential maximal function of f is defined as

$$N(f)(x) = \sup\{|f(y, t)| : (y, t) \in \Gamma(x)\}.$$

The area function of f is defined by

$$A(f) = \left(\int_{\Gamma(x)} |f(y,t)|^2 \frac{dydt}{t_1^2 t_2^2} \right)^{1/2}.$$

Theorem 4.4. *Let $0 < p < \infty$ and $\omega \in A_\infty^*$. There exists a constant $C > 0$ such that for any biharmonic function u on $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ satisfying, for any $x \in \mathbb{R}^2$, $\lim_{|t|\to\infty} u(x,t) = 0$, we have*

$$\int (N(u)(x))^p \omega(x) dx \le C \int (A(u)(x))^p \omega(x) dx.$$

We refer the reader to [**36**, Theorem 3] for the proof of the above result.

Write $u \in \mathcal{H}$ if u is a biharmonic function on $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ which satisfies, for any $x \in \mathbb{R}^2$, $\lim_{|t|\to\infty} u(x,t) = 0$.

Theorem 4.5. *Let $0 < p < 1$ and $p_1(\cdot), p_2(\cdot) \in \mathcal{B}$. There exists a constant $C > 0$ such that for any $u \in \mathcal{H}$, we have*

$$\|N(u)^p\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})} \le C \|A(u)^p\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})}.$$

Proof. Theorem 2.1 guarantees that there exists a $1 < q_0 < (p_1, p_2)_-$ such that $(p_1(\cdot)/q_0)', (p_2/q_0)' \in \mathcal{B}$. As $A_1^* \subset A_\infty^*$, Theorem 4.4 ensures that we can apply Theorem 3.2 to

$$\mathcal{F} = \{(|A(u)|^p, |N(u)|^p) : u \in \mathcal{H}\}.$$

By applying Theorem 3.2 with $p_0 = 1$, we find that

$$\|N(u)^p\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})} \le C \|A(u)^p\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)})}.$$

□

4.4 - *Ricci-Stein singular integrals*

We have to consider the mixed-norm Lebesgue spaces $(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})$ generated by a triplet of exponent functions $(p_1(\cdot), p_2(\cdot), p_3(\cdot))$ in this section. The mixed-norm Lebesgue space with variable exponent $(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})$ consists of all Lebesgue measurable function on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, f such that

$$\|f\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})} = \|\|\|f\|_{L^{p_1(\cdot)}}\|_{L^{p_2(\cdot)}}\|_{L^{p_3(\cdot)}} < \infty.$$

Notice that the results in Section 3 for $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ are also valid for $(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})$. Especially, we also have the extrapolation theory for $(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})$.

For simplicity, we just present the statement of the extrapolation theory for $(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})$. The proof follows from some standard modifications of the proof for [**21**, Theorem 3.2].

We restate some notations in order to present the statement for the extrapolation theory.

For any $r, s, t > 0$ and $z = (x, y, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, define $R(z, r, s, t) = B(x, r) \times B(y, s) \times B(u, t)$. Write $\mathcal{R}_3 = \{R(z, r, s, t) : z \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, s, r, t > 0\}$.

Definition 4.1. For $1 < p < \infty$, we say that a nonnegative locally integrable function $\omega \in A_p^{**}$ if

$$[\omega]_{A_p^{**}} = \sup_{R \in \mathcal{R}_3} \left(\frac{1}{|R|} \int_R \omega(z) dz \right) \left(\frac{1}{|R|} \int_R \omega(z)^{-\frac{1}{p-1}} dz \right)^{p-1} < \infty.$$

We say that a nonnegative measurable function $\omega \in A_1^{**}$ if

$$[\omega]_{A_1^{**}} = \sup_{R \in \mathcal{R}_3} \left(\frac{1}{|R|} \int_R \omega(z) dz \right) \operatorname*{ess\,sup}_{z \in R} \omega(z)^{-1} < \infty.$$

We write $A_\infty^{**} = \cup_{1 \le p < \infty} A_p^{**}$.

We now present the extrapolation theory for $(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})$.

Theorem 4.6. *Let* $p_i(\cdot) \in \mathcal{P}$, $i = 1, 2, 3$. *Given a family* $\mathcal{F}$, *suppose that for some* $0 < p_0 < \infty$ *and for every* $\omega_0 \in A_1^{**}$, *we have*

$$\int_{\mathbb{R} \times \mathbb{R}} f(x, y)^{p_0} \omega_0(x, y) dx dy \le C \int_{\mathbb{R} \times \mathbb{R}} g(x, y)^{p_0} \omega_0(x, y) dx dy \tag{14}$$

for any $(f, g) \in \mathcal{F}$ *where* C *depends only on* p_0 *and* $[\omega_0]_{A_1^{**}}$.

Suppose that there exists $p_0 \le q_0 < (p_1, p_2, p_3)_-$ *such that* $(p_i(\cdot)/q_0)' \in \mathcal{B}$, $i = 1, 2, 3$, *then,*

$$\|f\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})} \le C \|g\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})}, \quad (f, g) \in \mathcal{F}. \tag{15}$$

Next, we recall some preliminaries for the study of the Ricci-Stein singular integrals. This is a family of singular integrals associated with the Zygmund dilation

$$(x, y, u) \to (sx, ty, stu), \quad s, t > 0. \tag{16}$$

The singular integrals introduced by Ricci and Stein in [**34**] is defined as

$$Tf = K * f$$

where

$$K(x,y,z) = \sum_{k,j\in\mathbb{Z}} 2^{-2(k+j)}\psi_{j,k}\left(\frac{x}{2^k},\frac{y}{2^j},\frac{u}{2^{k+j}}\right).$$

The family of function $\{\psi_{j,k}\}$ satisfies the following conditions. There exists a sufficiently large positive integer N such that

1.

$$\|\psi_{j,k}\|_{\mathcal{S}_N} = \sup_{z\in\mathbb{R}^3}(1+|z|)^N\left(\sum_{\alpha,\beta,\gamma=0}^{N}|\partial^\alpha\partial^\beta\partial^\gamma\psi_{j,k}(z)|\right) < \infty.$$

2. For any fixed $x\in\mathbb{R}$ and all $\alpha,\beta\in\mathbb{N}$ with $\alpha,\beta<N$,

$$\int_{\mathbb{R}^2} y^\alpha u^\beta\psi_{j,k}(x,y,u)dydu = 0.$$

3. For any fixed $y\in\mathbb{R}$ and all $\alpha,\beta\in\mathbb{N}$ with $\alpha,\beta<N$,

$$\int_{\mathbb{R}^2} x^\alpha u^\beta\psi_{j,k}(x,y,u)dxdu = 0.$$

4. For any fixed $u\in\mathbb{R}$ and all $\alpha,\beta\in\mathbb{N}$ with $\alpha,\beta<N$,

$$\int_{\mathbb{R}^2} x^\alpha y^\beta\psi_{j,k}(x,y,u)dxdy = 0.$$

To apply the extrapolation theory, we need the weighted norm inequalities for T. The weighted norm inequalities for T relies on weight associated with Zygmund dilation (16). Let $\mathcal{R}_z$ be the class of rectangles in $\mathbb{R}^3$ whose sides are parallel to the axes and have side lengths of the form s, t and st.

Definition 4.2. For $1<p<\infty$, we say that a nonnegative locally integrable function $\omega\in A_p^z$ if

$$[\omega]_{A_p^z} = \sup_{R\in\mathcal{R}_z}\left(\frac{1}{|R|}\int_R\omega(z)dz\right)\left(\frac{1}{|R|}\int_R\omega(z)^{-\frac{1}{p-1}}dz\right)^{p-1} < \infty.$$

It is easy to see that for any $1 < p < \infty$,

$$A_p^{**} \subseteq A_p^z.$$

The following weighted norm inequalities for T are given in [**9**, Theorem 2.4].

Theorem 4.7. *Let $1 < p < \infty$ and $\omega \in A_p^z$. The Ricci-Stein singular integral T is bounded on $L^p(\omega)$.*

With the above result, we are now ready to apply the extrapolation theory to obtain the boundedness of T on $(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})$.

Theorem 4.8. *Let $p_i \in \mathcal{B}$, $i = 1, 2, 3$. The Ricci-Stein singular integral T can be extended to be a bounded operator on $(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})$*

Proof. For any $1 < p < \infty$, since $A_1^{**} \subset A_p^{**} \subset A_p^z$, Theorem 4.7 asserts that for any $\omega \in A_1^{**}$, we have

$$\int |Tf(x)|^p \omega(x) dx \leq \int |f(x)|^p \omega(x) dx.$$

Furthermore, Theorem 2.1 yields a $1 < q_0 < (p_1, p_2, p_3)_-$ such that $(p_i/q_0)' \in \mathcal{B}$, $i = 1, 2, 3$. We apply Theorem 4.6 with $p_0 = q_0$ on the set

$$\mathcal{F} = \{(Tf, f) : f \in C_c(\mathbb{R}^3)\}.$$

Consequently, Theorem 4.6 yields a constant $C > 0$ such that for any $f \in C_c(\mathbb{R}^3)$,

$$\|Tf\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})} \leq C\|f\|_{(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})}. \tag{17}$$

Since $C_c(\mathbb{R}^3)$ is dense in $(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})$, (17) guarantees that T can be extended to be a bounded linear operator on $(L^{p_1(\cdot)}, L^{p_2(\cdot)}, L^{p_3(\cdot)})$. □

4.5 - *Characterizations of BMO and John-Nirenberg inequalities*

In this section, we obtain the John-Nirenberg inequalities for $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$ and the characterization of the function space of bounded mean oscillation BMO via $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$. The John-Nirenberg inequalities and the characterization of BMO in terms of general function spaces had been extended to Lebesgue spaces with variable exponent, Morrey spaces and Banach function spaces, see [**16**, **17**, **18**, **19**, **20**, **22**, **23**, **24**, **25**, **26**, **27**].

For any $x \in \mathbb{R}^2$ and $r > 0$, let $B_2(x,r) = \{y \in \mathbb{R}^2 : |x-y| < r\}$ and $\mathbb{B}_2 = \{B_2(x,r) : x \in \mathbb{R}^2, r > 0\}$.

Let f be a locally integrable function on $\mathbb{R}^2$. We say that f belongs to BMO if

$$\|f\|_{BMO} = \sup_{B\in\mathbb{B}_2} \frac{\|(f-f_B)\chi_B\|_{L^1}}{\|\chi_B\|_{L^1}} < \infty$$

where $f_B = \frac{1}{|B|}\int_B f(x)dx$.

It is well known that BMO can be defined via the L^p norm. That is,

$$BMO = \left\{ f \in L^1_{loc} : \sup_{B\in\mathbb{B}_2} \frac{\|(f-f_B)\chi_B\|_{L^p}}{\|\chi_B\|_{L^p}} < \infty \right\}.$$

Recently, the characterization of BMO had been extended to weighted Lebesgue spaces [**17**], Lebesgue spaces with variable exponent [**22**,**24**], Banach function spaces [**16**,**18**,**26**] and Morrey spaces [**23**].

One of the tool to develop the characterization of BMO on the above function spaces is the John-Nirenberg inequalities. We recall the John-Nirenberg inequality from [**13**, Theorem 7.1.6].

Theorem 4.9. *There exist constants $C_1, C_2 > 0$ such that for any $\gamma > 0$ and any $B \in \mathbb{B}_2$,*

$$|\{x \in B : |f(x) - f_B| > \gamma\}| \leq C_1 e^{-\frac{C_2\gamma}{\|f\|_{BMO}}} |B|, \quad f \in BMO\backslash\mathcal{C}$$

where $\mathcal{C}$ denotes the set of constant functions.

The reader is referred to [**19**] for the John-Nirenberg inequalities on Lebesgue spaces with variable exponent and [**20**] for vector-valued John-Nirenberg inequalities.

The following theorem is an extension of the John-Nirenberg inequalities to $(L^{p_1(\cdot)}, L^{p_2(\cdot)})$.

Theorem 4.10. *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{B}$. There exist constants $C, C_1 > 0$ such that for any $\gamma > 0$, $f \in BMO\backslash\mathcal{C}$ and $B \in \mathbb{B}_2$, we have*

$$\|\chi_{\{x\in B:|f(x)-f_B|\geq\gamma\}}\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \leq C e^{-\frac{C_1\gamma}{\|f\|_{BMO}}} \|\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}.$$

Proof. In view of [**17**, (3.2)], there exist constants $D, E > 0$ such that for any $\omega \in A_1 \subset A_\infty$ and $B \in \mathbb{B}_2$, we have

$$\omega(\{x \in B : |f(x) - f_B| \geq \gamma\}) \leq D e^{-\frac{E\gamma}{\|f\|_{BMO}}} \omega(B).$$

Since $A_1^* \subset A_1$, the above inequality is also valid for $\omega \in A_1^*$. More precisely, there exist constants $D, E > 0$ such that for any $\omega \in A_1^*$ and $B \in \mathbb{B}_2$, we have

$$\int \chi_{\{x\in B:|f(x)-f_B|\geq\gamma\}}(y)\omega(y)dy \leq De^{-\frac{E\gamma}{\|f\|_{BMO}}} \int\limits_B \omega(y)dy.$$

Since $p_1(\cdot), p_2(\cdot) \in \mathcal{B}$, Theorem 2.1 guarantees that there exists a $1 < q_0 < (p_1, p_2)_-$ such that $(p_1/q_0)', (p_2/q_0)' \in \mathcal{B}$. We are allowed to apply Theorem 3.2 with $p_0 = 1$ on

$$\mathcal{F} = \{(\chi_{\{x\in B:|f(x)-f_B|\geq\gamma\}}, De^{-\frac{E\gamma}{\|f\|_{BMO}}}\chi_B) : B \in \mathbb{B}_2\}.$$

Theorem 3.2 yields constants $C, C_1 > 0$ such that for any $\gamma > 0$, $f \in BMO\backslash\mathcal{C}$ and $B \in \mathbb{B}_2$, we have

$$\|\chi_{\{x\in B:|f(x)-f_B|\geq\gamma\}}\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} \leq Ce^{-\frac{C_1\gamma}{\|f\|_{BMO}}}\|\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}.$$

□

Theorem 4.11. *Let $p_1(\cdot), p_2(\cdot) \in \mathcal{B}$. The norm*

$$\|f\|_{BMO_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}} = \sup_{B\in\mathbb{B}} \frac{\|(f - f_B)\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}}{\|\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}}$$

and $\|\cdot\|_{BMO}$ are mutually equivalent on BMO.

Proof. The Hölder inequality (7) ensures that for any $f \in BMO$ and $B \in \mathbb{B}_2$,

$$\frac{1}{|B|}\|(f - f_B)\chi_B\|_{L^1} \leq \frac{1}{|B|}\|(f - f_B)\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}\|\chi_B\|_{(L^{p_1'(\cdot)},L^{p_2'(\cdot)})}.$$

For any $B \in \mathbb{B}_2$, there exists a $R \in \mathcal{R}$ such that $B \subseteq R$ and

$$|B| \leq |R| \leq 2|B|.$$

Therefore, (9) yields a constant $C > 0$ such that for any $B \in \mathbb{B}_2$, we have

$$\begin{aligned}\|\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}\|\chi_B\|_{(L^{p_1'(\cdot)},L^{p_2'(\cdot)})} &\leq \|\chi_R\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}\|\chi_R\|_{(L^{p_1'(\cdot)},L^{p_2'(\cdot)})} \\ &\leq C|R| \leq 2C|B|.\end{aligned}$$

Consequently,

$$\frac{\|(f-f_B)\chi_B\|_{L^1}}{|B|} \leq C\frac{\|(f-f_B)\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}}{\|\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}} \tag{18}$$

for some $C>0$. By taking supremum over all $B\in\mathbb{B}_2$, we obtain

$$\begin{aligned}\|f\|_{BMO} &= \sup_{B\in\mathbb{B}_2}\frac{\|(f-f_B)\chi_B\|_{L^1}}{|B|}\\ &\leq C\sup_{B\in\mathbb{B}_2}\frac{\|(f-f_B)\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}}{\|\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}} = C\|f\|_{BMO_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}}.\end{aligned}$$

Next, for any $f\in BMO$ and $B\in\mathbb{B}_2$,

$$\begin{aligned}&\|(f-f_B)\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}\\ &\leq \|\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} + \sum_{k=0}^{\infty}2^{k+1}\|\chi_{\{x\in B:2^k<|f(x)-f_B|\leq 2^{k+1}\}}\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}.\end{aligned}$$

Theorem 4.9 ensures that

$$\begin{aligned}&\|(f-f_B)\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}\\ &\leq \|\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})} + C\sum_{k=0}^{\infty}2^{k+1}e^{-\frac{C_1 2^{k+1}}{\|f\|_{BMO}}}\|\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}.\end{aligned}$$

Since

$$\sum_{k=0}^{\infty}2^{k+1}e^{-\frac{C_1 2^{k+1}}{\|f\|_{BMO}}} \leq C\int_0^{\infty}\exp\Big(-\frac{C_1 s}{\|f\|_{BMO}}\Big)ds \leq C\|f\|_{BMO},$$

we obtain

$$\frac{\|(f-f_B)\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}}{\|\chi_B\|_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}} \leq C\|f\|_{BMO}$$

for some $C>0$ independent of $B\in\mathbb{B}_2$ and $f\in BMO$. By taking supremum over $B\in\mathbb{B}_2$, we have $\|f\|_{BMO_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}} \leq C\|f\|_{BMO}$.

Therefore, $\|\cdot\|_{BMO_{(L^{p_1(\cdot)},L^{p_2(\cdot)})}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent on BMO. □

In particular, the above result also gives the characterization of BMO in terms of mixed-norm Lebesgue spaces (L^{p_1},L^{p_2}) whenever $1<p_1,p_2<\infty$.

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