Riv. Mat. Univ. Parma, Vol. **9** (2018), 45-51

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Riemannian G-manifolds of constant negative curvature whose all orbits are principal

Abstract. We give a topological classification on Riemannian G-manifolds of constant negative curvature and their orbits, under the condition that all orbits are principal.

Keywords. Riemannian manifold, Lie group, Isometry.

Mathematics Subject Classification (2010): 53C30, 57S25.

1 - Introduction

If M is a Riemannian manifold and G is a closed and connected subgroup of the Lie group of the isometries of M, then M is called a Riemannian G-manifold. It is well known that for a Riemannian G-manifold M there is a unique maximal orbit type called the principal orbits and the other orbits are called singular orbits (see [**5**]). The union of all principal orbits is an open and dense subset of M. Most studied families of Riemannian G-manifolds are those which are homogeneous. These are Riemannian G-manifolds with transitive action of G on M. Topology and geometry of homogeneous Riemannian manifolds are for the most part well understood. The next most studied family are those which the dimension of the orbit space, that is called the cohomogeneity of the G-manifold, is one. Also, there are many results about topological properties of Riemannian G-manifolds of cohomogeneity two or three. But, in the general case, where the cohomogeneity is considered to be arbitrary, the problem of topological classification on Riemannian G-manifolds and their orbits is an open problem. To study of the G-manifolds of arbitrary cohomogeneity, it seems better to start by the special case where all orbits are considered

Received: February 24, 2018; accepted in revised form: August 7, 2018.

to be principal. For symmetric spaces of noncompact type $\frac{G}{K}$, the orbits of a cohomogeneity one action without singular orbit form a Riemannian foliation. Furthermore, when $\frac{G}{K}$ is irreducible, such actions have been completely classified up to orbit equivalence ([**2**]). It is proved in [**21**] that if M^n is a cohomogeneity one Riemannian G-manifold without singular orbit then either M is simply connected or it is diffeomorphic to $R^k \times T^{n-k}$, for some positive integer k, and each orbit is diffeomorphic to $R^k \times T^{n-k-1}$. There are some results about cohomogeneity two Riemannian G-manifolds without singular orbit in [**16**], and cohomogeneity two Riemannian manifolds of constant negative curvature have been studied in [**15**]. To extend the results of [**15**] and [**16**], we consider in the present paper, a Riemannian G-manifold of constant negative curvature and arbitrary cohomogeneity whose all orbits are principal.

2 - Preliminaries

Throughout the following, M is a complete and connected Riemannian manifold of negative curvature and G is a closed and connected subgroup of the Lie group of all isometries of M. If M is not simply connected, then $\widetilde{M}$ will denote the universal covering manifold of M endowed with the pulled back metric, and $\kappa : \widetilde{M} \to M$ will denote the covering projection. We will use the symbol Δ to indicate the decktransformation group of the covering map $\kappa : \widetilde{M} \to M$. Note that the Lie group G admits always a connected covering $\widetilde{G}$ which acts on $\widetilde{M}$ by isometries. The covering map $\pi : \widetilde{G} \to G$ is such that $\pi(g')(x) = \kappa(g'(y))$ for all $x \in M$, $g' \in \widetilde{G}$, $y \in \kappa^{-1}(x)$. Moreover, Δ centralizes $\widetilde{G}$, so it maps $\widetilde{G}$-orbits onto $\widetilde{G}$-orbits (see [**5**] pages 63 and 64).

Remark 2.1. If K is a maximal compact subgroup of a connected Lie group G, then there is a solvable Lie group H which acts transitively on $\frac{G}{K}$.

Proof. We have $G = S.R$ such that R is a maximal solvable and normal subgroup of G and S is a semisimple subgroup (Levi decomposition). S can be decomposed as $S = S_N.S_C$ where S_N and S_C are product of simple noncompact and simple compact subgroups, respectively. Since K is maximal compact in G, then $K = K_N.S_C.K_R$ where K_N and K_R are maximal compact in S_N and R. By Iwasawa decomposition, we have $S_N = B.K_N$ such that B is a solvable group. Since $R = R.K_R$, then we have:

$$G = R.S = R.K_R.S_N.S_C = R.K_R.(B.K_N).S_C$$

$$= R.B.K_N.K_R.S_C = R.B.K$$

So, $R.B$ is a solvable Lie group and acts transitively on $\frac{G}{K}$. □

Remark 2.2. If M is a G-manifold and $x \in M$, the G-orbit containing x is denoted by $G(x) = \{gx : g \in G\}$. The set $G_x = \{g \in G : gx = x\}$ is called the stabilizer group of $x \in M$. Two orbits $G(p)$ and $G(q)$ are said to have the same orbit type if G_p and G_q are conjugate in G. This defines an equivalence relation among the orbits of G. We denote by $[G(p)]$ the corresponding equivalence class, which is called the orbit type of $G(p)$. We can impose a partial ordering on the set of all orbit types of the action of G on M by saying that $[G(p)] \leq [G(q)]$ if and only if G_q is conjugate in G to some subgroup of G_p. An orbit $G(p)$ is principal if and only if for each $q \in M$ the isotropy group G_p at p is conjugate in G to some subgroup of G_q. A non-principal orbit is called singular orbit. Therefore, if there is no singular orbit then for all $x \in M$, G_x is a maximal compact subgroup of G.

Theorem 2.1 ([**12**,**24**]).

1) *A homogeneous Riemannian manifold of negative curvature is simply connected.*
2) *A homogeneous Riemannian manifold M^n of nonpositive curvature is diffeomorphic to $T^p \times R^{n-p}$, $p \geq 0$.*

Remark 2.3. **(1)** If M^n is a connected and complete Riemannian manifold of constant negative curvature c, then its universal covering manifold is $H^n(c)$ (hyperbolic space). In the open disk model, $H^n(c)$ can be considered as the set $D = \{x \in R^n : |x| < 1\}$. S^{n-1}, the boundary of D in R^n is the infinity set of $H^n(c)$ denoted by $H^n(c)(\infty) = S^{n-1}$. If $z \in H^n(c)(\infty)$ and γ is a geodesic in $H^n(c)$ such that $lim_{t\to\infty}\gamma(t) = z$, then z is denoted by $\gamma(\infty)$. The collection of all geodesics γ with the property $\gamma(\infty) = z$ are asymptotic and are denoted by $[\gamma]$. If $z \in H^n(c)(\infty)$ then there is a foliation of hypersurfaces of $H^n(c)$, called *horospheres centered at* z, such that at each point $a \in M$, a horosphere S centered at z is perpendicular to a unique geodesic γ with $\gamma(\infty) = z$. It is well known that horospheres of $H^n(c)$ are isometric to R^{n-1}.

(2) By general theory of Riemannian manifolds of negative curvature (see [**3**]), if ϕ is an isometry of $H^n(c)$ then one the following is true:
ϕ has a fixed point in $H^n(c)$ (ϕ is elliptic),
ϕ translates a unique geodesic (ϕ is hyperbolic),
ϕ has a fixed point in $H^n(c)(\infty)$ (ϕ is parabolic).

(3) If ϕ is a parabolic isometry and $z \in H^n(c)(\infty)$ such that $\phi(z) = z$, then for all horospheres S centered at z, $\phi(S) = S$ ([**6**, Lemma 3]).

(4) If $\delta \in \Delta$ then δ is not elliptic (a decktransformation has no fixed point). So, δ translates a unique geodesic or it is parabolic.

3 - Results

If G and H are Lie groups acting on a Riemannian manifold M with the same orbits, then we say that G and H are orbit equivalent.

We will use Lemma 3.1 and Theorem 3.1 in the proof of Theorem 3.2. For more details about isometric actions on R^n, we refer to [**18**].

Lemma 3.1. *Let $x \in R^n$ and let G be a closed, connected and noncompact subgroup of $Iso(R^n)$. Then one of the following is true:*

(1) *There is a positive integer l such that, $G(\circ) = R^l$ ($\circ$ is the origin of R^n), and each orbit $G(x)$ splits as $G(x) = M_x \times R^l$, $M_x \subset R^{n-l}$.*

(2) *There are nonnegative integers d, e such that $d + e = n$ and G is orbit equivalent to a subgroup of $So(d) \times R^e$.*

Proof. The proof of Lemma 2.6 in [**18**] works. Note that in the proof of the mentioned lemma, the parallel distribution D is defined on R^n. Since for $x = \circ$ the action of the members of G on x is equal to the action of the translation part of G, we have $G(\circ) = R^l$. □

Theorem 3.1. *If G is a closed, noncompact and connected subgroup of $Iso(R^n)$ such that all orbits are principal, then one of the following is true:*

(1) *There is a unique totally geodesic orbit.*

(2) *All orbits are totally geodesic.*

Proof. Consider two cases (1) and (2), in Lemma 3.1.

Case 1. Since all orbits are principal, then for all $x \in R^n$, $\dim G(x) = \dim G(o)$. So, $\dim M_x = 0$ and $G(x) \simeq R^l$ for all $x \in R^n$.

Case 2. If $d = 0$ then all orbits are isometric to R^e.

If $d \neq 0$ then for $x = (o, o) \in R^d \times R^e$, we have $G(x) \simeq R^e$, and for $x = (x_1, x_2) \in R^d \times R^e$, $x_1 \neq o$, the orbit $G(x)$ is a submanifold of $S^{d-1}(r) \times R^e$, which is not euclidean (it is a generalized helicoid). Since the orbits are connected, then $d \neq 2$. So, there is a unique euclidean orbit. □

Remark 3.1 ([**8**]). If M is nonsimply connected of negative curvature and there is a geodesic γ in $\widetilde{M}$ such that $\Delta(\gamma) = \gamma$, then Δ is isomorphic to $(Z, +)$.

Theorem 3.2. *Let M^n, $n \geq 2$, be a complete and connected Riemannian G-manifold of constant negative curvature such that G is not trivial and there is no singular orbit. Then, either all orbits are flat or one of the following is true:*

(a) *All orbits are diffeomorphic to* $R^k \times T^s$ *for some non-negative integers* k, s. M *is diffeomorphic to* R^n.

(b) *All orbits are diffeomorphic to* S^1. M *is diffeomorphic to* $\frac{R^n}{Z}$ *for some action of* Z *on* R^n.

(c) *All orbits are diffeomorphic to* $R^k \times T^s$ *for some non-negative integers* k, s. M *is diffeomorphic to* $\frac{R^n}{Z^s}$ *for some action of* Z^s *on* R^n.

Proof. We use the symbols mentioned in the first paragraph of the preliminaries. Since M has constant negative curvature c, then we have $\widetilde{M} = H^n(c)$. If M is simply connected then $M = H^n(c)$, so M is diffeomorphic to R^n. For each point $x \in M$ the orbit $G(x)$ is diffeomorphic to $\frac{G}{G_x}$. Since there is no singular orbit, by the last lines of Remark 2.2, G_x is a maximal compact subgroup of G. So, by Corollary 2.1, $G(x)$ is a solvmanifold and by the main theorem of [**11**], it must be diffeomorphic to $R^k \times T^s$ for some $m, s \geq 0$. This is part (a) of the theorem.

Now, suppose that M is not simply connected. By the main theorem of [**22**], one of the following is true:

(1) There is a zero dimensional $\widetilde{G}$-orbit in $H^n(c)$.

(2) There is a unique totally geodesic $\widetilde{G}$-orbit.

(3) There is a point $z \in H^n(c)(\infty)$ such that all orbits are included in horospheres centered at z.

We consider each case separately.

(1) There is also a zero dimensional G-orbits in M. Since by assumptions all orbits are principal, all G-orbits must be zero dimensional. Then, the action of G must be trivial, which is a contradiction.

(2) Let $N = \widetilde{G}(x)$ be the unique totally geodesic orbit of $H^n(c)$. We get from the uniqueness of N that $\Delta(N) = N$ (because, if $\delta \in \Delta$ then from the fact that the members of Δ map orbits to orbits, $\delta(N)$ is also a totally geodesic $\widetilde{G}$-orbit). Now, put $D = \kappa(N)$. D is a G-orbit in M. Since $\Delta(N) = N$, then $D = \frac{N}{\Delta}$. If $\dim N \geq 2$ then D would be a homogeneous Riemannian manifold of negative curvature and by Theorem 2.1, D must be simply connected. So $\Delta (= \pi_1(D))$ is trivial and M is simply connected.

If $\dim N = 1$ then N is equal to the image of a geodesic λ and we get from $\Delta(N) = N$ and Remark 3.1 that $\Delta = Z$. Also, $\frac{\lambda}{\Delta}$ is a G-orbit in M which is diffeomorphic to $\frac{R}{Z} = S^1$. Since $\widetilde{M}$ is diffeomorphic to R^n and all orbits are diffeomorphic to each other, we get part (b) of the theorem.

(3) First note that if there is a $\delta \in \Delta$ and a unique geodesic γ such that $\delta(\gamma) = \gamma$, then we get from the uniqueness of γ that γ is a $\widetilde{G}$-orbit. So, we get part (b) of the theorem in a similar way as (2). Thus, by Remark 2.3(4), we can

suppose that all members of Δ are parabolic. Since each $\delta \in \Delta$ maps orbits to orbits, then δ leaves invariant the horosphere foliation centered at z. Thus, by Remark 2.3(3), $\Delta(S) = S$. Consider a horosphere S in the mentioned foliation. S is a $\widetilde{G}$-manifold without singular orbit. Since by Remark 2.3(1), S is isometric to R^{n-1} then by Theorem 3.1, either all orbits are Euclidean or there is only one Euclidean orbit in S. By assumptions of the theorem, we can suppose that there exists a unique Euclidean orbit E in $S = R^{n-1}$. Since $\Delta(S) = S$, we get from the uniqueness of E in S that $\Delta(E) = E$. Now, put $D = \kappa(E)$. D is a G-orbit which is flat. Thus, by Theorem 2.1(2), D is diffeomorphic to $R^k \times T^s$ for some nonnegative integers k and s. Since $\pi_1(M) = \Delta = \pi_1(D)$, we get part (c) of the theorem. □

Acknowledgments. The author would like to thank the referee and the editors for their time spent on reviewing the manuscript.

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