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A multiplicity result for a fractional p-Laplacian problem without growth conditions

Abstract. Using an abstract critical point result due to Ricceri and combining a truncation argument with a Moser-type iteration, we establish the existence of at least three bounded solutions for a fractional p-Laplacian problem depending on two parameters and involving nonlinearities with arbitrary growth.

Keywords. Fractional p-Laplacian, arbitrary growth, multiple solutions, Moser-type iteration.

Mathematics Subject Classification (2010): 35A15, 35R11, 45G05.

1 - Introduction

In this paper we investigate the existence of multiple solutions for the following fractional p-Laplacian problem

$$\begin{cases} (-\Delta)^s_p u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded open set, $s \in (0,1)$, $p \in (1,\infty)$, λ and μ are real parameters, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Caratheodory functions. Here $(-\Delta)^s_p$ is the fractional p-Laplacian operator defined, for any $u : \mathbb{R}^N \to \mathbb{R}$ smooth enough, by the formula

$$(-\Delta)^s_p u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy, \quad x \in \mathbb{R}^N,$$

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consistent, up to some normalization constant depending upon N and s, with the definition of fractional Laplacian $(-\Delta)^s$ in the case $p = 2$.

In the last decade a great interest has been devoted to the study of fractional operators, both for the pure mathematical research and in view of their relevance in many fields, due to the fact that they appear in many different and several contexts such as phase transitions, crystal dislocations, anomalous diffusions, minimal surfaces, fluid dynamics, just to name a few. For more details on this subject we refer the interested reader to [**11**] and [**22**].

In particular way, in recent years, non-local problems involving the fractional p-Laplacian operator have received the attention of many mathematicians. Franzina and Palatucci [**14**] and Lindgren and Linqvist [**18**] studied the eigenvalue problem $(-\Delta)^s_p u = \lambda |u|^{p-2}u$, proving some properties of the first eigenvalue and of the higher order eigenvalues. Pucci et al. [**26**] obtained a multiplicity result for the following nonhomogeneous Schrödinger-Kirchhoff equation

$$M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy\right)(-\Delta)^s_p u + V(x)|u|^{p-2}u = f(x,u) + g(x) \text{ in } \mathbb{R}^N,$$

where f satisfies the Ambrosetti-Rabinowitz condition, V is an external potential, and $g \in L^{\frac{p}{p-1}}(\mathbb{R}^N)$ is a perturbation term. Iannizzotto et al. [**15**] established by means of variational and topological methods, some existence and multiplicity results for subcritical fractional p-Laplacian problems with homogeneous Dirichlet boundary conditions of the type

$$\begin{cases} (-\Delta)^s_p u = h(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Mosconi et al. [**24**] dealt with the existence of nontrivial solutions for a fractional Brezis-Nirenberg problem. In [**2**] the author established via a variant of the Fountain Theorem, the existence of infinitely many solutions for a fractional p-Laplace equation in $\mathbb{R}^N$ with a potential $V(x)$ allowed to be sign changing, and a p-superlinear nonlinearity. Di Castro et al. [**9**] proved a general Harnack inequality for minimizers of nonlocal, possibly degenerate, integro-differential operators, whose model is the fractional p-Laplacian operator; see also [**10, 17**] for related regularity results. In [**4**] the author investigated the existence of nontrivial solutions for a fractional p-Laplacian problem via a generalization of the Struwe-Jeanjean monotonicity trick.

Motivated by the above papers, in the present work we aim to study the multiplicity of weak solutions of a non-local parametric problem in presence of perturbations with arbitrary growth. It is worth recalling that when f is superlinear at zero and sublinear at infinity, and $sp < N$, the existence of three critical points can be obtained by using the abstract result in [**28**]; see for instance [**21**, **23**] for related results in non-local setting. The situation is completely different if g has a super-critical or arbitrary growth, because the corresponding Euler-Lagrange functional associated with problem (1) is not well-defined on the fractional Sobolev space $W_0^{s,p}(\Omega)$, and we cannot apply directly known variational techniques. To overcome this difficulty we borrow some ideas developed in [**6**,**12**,**16**,**19**,**30**] in which the authors established some multiplicity results for boundary value problems with nonlinearities having critical, super-critical and arbitrary growth; see also [**7**, **13**] for super-critical problems in $\mathbb{R}^N$. More precisely, we first consider a truncated problem [**27**] with subcritical growth, and then we apply the abstract critical point in [**28**] to get a multiplicity of solutions u_K for truncated problem (P_K). At this point we use an appropriate Moser-iteration scheme [**20**, **25**] to prove that, if K is chosen in a suitable way, the solutions of the truncated problem also verify the original problem (1). We mention that in a recent paper [**5**] a similar argument combined with a variant of the extension method [**8**] obtained by the author in [**1**, **3**], has been used to prove the existence of multiple periodic solutions. Anyway, the iteration used in [**5**] does not work when we have an arbitrary perturbation. We also point out that due to the presence of non-local operator $(-\Delta)_p^s$, which is not linear when $p \neq 2$, the approach developed in [**5**] is not so easy to adapt in our setting, and more accurate arguments are needed.

Before stating our main result we introduce the main assumptions on the nonlinearities appearing in (1). Let us denote by $\mathcal{A}$ the class of Caratheodory functions $h : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{|t| \leq M} |h(x,t)| \in L^\infty(\Omega) \text{ for any } M > 0.$$

We assume that the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ belongs to the class $\mathcal{A}$ and satisfies the following assumptions:

$(f1)$ $\lim_{|t|\to 0} \dfrac{f(x,t)}{|t|^{p-1}} = 0$ uniformly in $x \in \Omega$;

$(f2)$ $\lim_{|t|\to \infty} \dfrac{f(x,t)}{|t|^{p-1}} = 0$ uniformly in $x \in \Omega$;

$(f3)$ $\sup_{t\in\mathbb{R}} \inf_{x\in\Omega} f(x,t) > 0$.

Then we are able to prove the following result.

Theorem 1.1. *Let $f \in \mathcal{A}$ be a function verifying (f1)-(f3), and let*

$$\theta := \frac{1}{p} \inf\left\{ \frac{\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy}{\int_\Omega F(x,u)dx} : \int_\Omega F(x,u)dx > 0 \right\}.$$

Then, for each compact interval $[a,b] \subset (\theta, \infty)$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$ and for every $g \in \mathcal{A}$, there exists $\mu^* > 0$ such that for every $\mu \in [0, \mu^*]$, problem (1) has at least three weak solutions whose norms in $W_0^{s,p}(\Omega)$ are less than ρ.

We underline that the parameters θ and ρ appearing in Theorem 1.1 depend on f but not on the particular choice of g. This enables to truncate the perturbation g in an appropriate way and apply the abstract multiplicity result in [**28**] to obtain the existence of at least three solutions for the truncated problem. Applying a suitable iteration technique, we will prove that these functions are bounded and, as a consequence, they will be solutions to (1).

The paper is organized as follows. In Section 2 we recall some basic facts concerning the fractional Sobolev spaces. In Section 3 we provide the proof of Theorem 1.1 and we present an application.

2 - Preliminaries

In this section we collect some useful results related to the fractional Sobolev spaces. For more details on this topic, we derive the interested reader to [**11**,**22**].

Let $s \in (0,1)$ and $p \in (1,\infty)$. We denote by

$$p_s^* := \begin{cases} \dfrac{Np}{N-sp} & \text{if } N > sp \\ \infty & \text{if } N \leq sp \end{cases}$$

the critical Sobolev exponent, and we set

$$\bar{p}_s^* := \begin{cases} p_s^* & \text{if } N > sp \\ \text{any positive number} > p & \text{if } N \leq sp. \end{cases}$$

Let $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function. We say that u belongs to the space $W^{s,p}(\mathbb{R}^N)$ if $u \in L^p(\mathbb{R}^N)$ and

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dxdy < \infty.$$

Then $W^{s,p}(\mathbb{R}^N)$ is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left[\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\right]^{1/p}.$$

We introduce the following closed linear subspace (see [**22**, **29**] for the case $p = 2$)

$$W_0^{s,p}(\Omega) = \left\{u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\},$$

which can be equivalently renormed by setting

$$\|u\|_{W_0^{s,p}(\Omega)} = [u]_{W^{s,p}(\mathbb{R}^N)}.$$

We denote by $(W^{-s,p'}(\Omega), \|\cdot\|_{W^{-s,p'}(\Omega)})$ the dual space of $(W_0^{s,p}(\Omega), \|\cdot\|_{W_0^{s,p}(\Omega)})$. Here p' is the conjugate exponent of p, that is $p' = \frac{p}{p-1}$.

In what follows, we recall some useful properties concerning the fractional Sobolev space $W_0^{s,p}(\Omega)$.

Theorem 2.1. [**11**] *Let $s \in (0,1)$ and $p \in (1,\infty)$. Then $(W_0^{s,p}(\Omega), \|\cdot\|_{W_0^{s,p}(\Omega)})$ is a separable and reflexive Banach space.*

Theorem 2.2. [**11**] *Let $s \in (0,1)$ and $p \in [1,\infty)$.*

- *If $N > sp$, then the embedding $W_0^{s,p}(\Omega) \subset L^r(\Omega)$ is continuous for any $r \in [1, p_s^*]$, and compact for $r \in [1, p_s^*)$.*
- *If $N = sp$, then $W_0^{s,p}(\Omega)$ is continuously embedded into $L^r(\Omega)$ for any $r \in [p, \infty)$.*
- *If $N < sp$, then $W_0^{s,p}(\Omega)$ is continuously embedded into $C^{0,\frac{sp-N}{p}}(\Omega)$.*

Now we provide the notion of weak solution for equations driven by the fractional p-Laplacian operator with homogeneous Dirichlet boundary conditions.

Definition 2.1. Let $f \in W^{-s,p'}(\Omega)$. We say that $u \in W_0^{s,p}(\Omega)$ is a weak solution to (1) if

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} (v(x) - v(y))\, dxdy = \langle f, v\rangle$$

holds for any $v \in W_0^{s,p}(\Omega)$. Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{s,p}(\Omega)$ and its dual $W^{-s,p'}(\Omega)$.

3 - Proof of Theorem 1.1

In the present section, we discuss the existence and the multiplicity of bounded solutions to (1). To achieve our purpose, we will apply an abstract multiplicity theorem and we use a truncation argument with a Moser iteration technique. Firstly we recall some useful notations. If X is a real Banach space, we denote by $\mathcal{W}_X$ the class of all functionals $\Phi : X \to \mathbb{R}$ having the following property: if $\{u_j\}_{j\in\mathbb{N}}$ is a sequence in X converging weakly to $u \in X$ and

$$\liminf_{j\to\infty} \Phi(u_j) \leq \Phi(u),$$

then $\{u_j\}_{j\in\mathbb{N}}$ has a subsequence converging strongly to u.
Now we state the following fundamental result due to Ricceri [**28**].

Theorem 3.1. [**28**] *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a coercive, sequentially weakly lower semicontinuous C^1-functional belonging to $\mathcal{W}_X$, bounded on each bounded subset of X, and whose derivative admits a continuous inverse on X^*; and $J : X \to \mathbb{R}$ a C^1-functional with compact derivative. Assume that Φ has a strict local minimum $\bar{u}$ with $\Phi(\bar{u}) = J(\bar{u}) = 0$. Finally, setting*

$$\alpha := \max\left\{0, \limsup_{\|u\|\to\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to\bar{u}} \frac{J(u)}{\Phi(u)}\right\}$$

$$\beta := \sup_{u\in\Phi^{-1}(0,\infty)} \frac{J(u)}{\Phi(u)},$$

assume that $\alpha < \beta$. Then, for each compact interval $[a, b] \subseteq (\frac{1}{\beta}, \frac{1}{\alpha})$ there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1-functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the equation

$$\Phi'(u) = \lambda J'(u) + \mu\Psi'(u)$$

has at least three solutions in X whose norms are less than ρ.

In order to apply the above theorem, we define the following functionals

$$\Phi(u) = \frac{1}{p}\|u\|^p_{W^{s,p}_0(\Omega)} \text{ and } J(u) = \int_\Omega F(x, u)dx.$$

for any $u \in W^{s,p}_0(\Omega)$. Using Theorem 2.1 and Theorem 2.2, and recalling the definition of $\bar{p}^*_s$, we deduce that $W^{s,p}_0(\Omega)$ is a reflexive Banach space which is continuously embedded into $L^{\bar{p}_s^*}(\Omega)$. From the assumptions on f, it is clear that J is a continuously Gateaux differentiable functional with compact derivative. Now we prove the following useful result.

Lemma 3.1. *Assume that $f \in \mathcal{A}$ satisfies $(f1)$-$(f2)$. Then we have*

$$\limsup_{\|u\|_{W_0^{s,p}(\Omega)}\to\infty} \frac{J(u)}{\Phi(u)} \leq 0 \ \textit{and} \ \limsup_{\|u\|_{W_0^{s,p}(\Omega)}\to 0} \frac{J(u)}{\Phi(u)} \leq 0. \tag{2}$$

Proof. Fix $\varepsilon > 0$. Using assumption $(f1)$ there exists $\delta_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon |t|^{p-1} \text{ for a.e. } x \in \Omega, \text{ for any } |t| \leq \delta_\varepsilon. \tag{3}$$

Now we take $r \in (p, p_s^*)$. From assumption $(f2)$ we deduce that there exists $M_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon |t|^{r-1} \text{ for a.e. } x \in \Omega, \text{ for any } |t| \geq M_\varepsilon. \tag{4}$$

Putting together (3) and (4) we can find $C_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon |t|^{p-1} + C_\varepsilon |t|^{r-1} \text{ for a.e. } x \in \Omega, \text{ for any } t \in \mathbb{R}. \tag{5}$$

Taking into account (5) and Theorem 2.2 we get

$$\left| \int_\Omega F(x,u)dx \right| \leq C'\varepsilon \|u\|^p_{W_0^{s,p}(\Omega)} + C''_\varepsilon \|u\|^r_{W_0^{s,p}(\Omega)} \tag{6}$$

from which we obtain (being $r > p$) that

$$\limsup_{\|u\|_{W_0^{s,p}(\Omega)}\to 0} \frac{J(u)}{\Phi(u)} \leq C'''\varepsilon. \tag{7}$$

From the arbitrariness of ε we can infer that the first relation of limit in (2) holds true. Now we prove the second relation of limit in (2). Using $(f2)$ there exists $M_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon |t|^{p-1} \text{ for a.e. } x \in \Omega, \text{ for any } |t| \geq M_\varepsilon. \tag{8}$$

Since $f \in \mathcal{A}$, there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \leq C_\varepsilon \text{ for a.e. } x \in \Omega, \text{ for any } |t| \leq M_\varepsilon. \tag{9}$$

Putting together (8) and (9) we find

$$|f(x,t)| \leq C_\varepsilon + \varepsilon |t|^{p-1} \text{ for a.e. } x \in \Omega, \text{ for any } t \in \mathbb{R}. \tag{10}$$

Therefore, using (10) and Theorem 2.2 we get

$$\left|\int_\Omega F(x,u)dx\right| \leq C'_\varepsilon \|u\|_{W_0^{s,p}(\Omega)} + C''\varepsilon\|u\|^p_{W_0^{s,p}(\Omega)},$$

and this implies that

$$\limsup_{\|u\|_{W_0^{s,p}(\Omega)}\to\infty} \frac{J(u)}{\Phi(u)} \leq C'''\varepsilon \tag{11}$$

for any $\varepsilon > 0$. This ends the proof of lemma. □

As a consequence of Lemma 3.1, we know that $\alpha = 0$. By assumption $(f3)$, it follows that $\beta > 0$ and that $\beta = \frac{1}{\theta}$. Then we are in the position to apply Theorem 3.1. Therefore, for each fixed compact interval $[a,b] \subset (\theta, \infty)$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$ and every C^1-functional $\Psi : W_0^{s,p}(\Omega) \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0,\delta]$ the equation

$$\Phi'(u) = \lambda J'(u) + \mu\Psi'(u)$$

has at least three solutions in $W_0^{s,p}(\Omega)$ whose norms are less than ρ.

Take $[a,b] \subset (\theta,\infty)$ and $\lambda \in [a,b]$. Since $f \in \mathcal{A}$ verifies $(f1)$ and $(f2)$, there exists $C_f > 0$ such that

$$|f(x,t)| \leq C_f|t|^{p-1} \text{ for a.e. } x \in \Omega, \text{ for any } t \in \mathbb{R}. \tag{12}$$

Let $g \in \mathcal{A}$ and we define

$$g_K(x,t) = \begin{cases} g(x,-K) & \text{if } t < -K \\ g(x,t) & \text{if } |t| \leq K \\ g(x,K) & \text{if } t > K, \end{cases} \tag{13}$$

where $K > 0$ will be determined later.

Let us denote by

$$G(x,t) = \int_0^t g(x,\tau)d\tau \text{ and } G_K(x,t) = \int_0^t g_K(x,\tau)d\tau,$$

and we set

$$\Psi(u) = \int_\Omega G_K(x,u)dx.$$

Since $g \in \mathcal{A}$, Ψ is continuously Gateaux differentiable with compact derivative. Hence we can find $\delta_0 > 0$ such that, for any $\mu \in [0, \delta_0]$, the following truncated problem

$$(P_K) \qquad \begin{cases} (-\Delta)_p^s u = \lambda f(x,u) + \mu g_K(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

has at least three weak solutions $u_i \in W_0^{s,p}(\Omega)$ such that $\|u_i\|_{W_0^{s,p}(\Omega)} < \rho$, for any $i = 1, 2, 3$.

In order to prove that each u_i is also a solution of the original problem (1), we aim to show that for any μ small enough we have

$$\|u_i\|_{L^\infty(\Omega)} \le K \text{ for any } i = 1, 2, 3. \tag{14}$$

Let us denote by u one of these solutions. Set $C_g = \operatorname{ess\,sup}_{x\in\Omega} | \sup_{|t|\le K} g(x,t)|$. Then it is clear that

$$|g_K(x,t)| \le C_g \text{ for a.e. } x \in \Omega, \text{ for any } t \in \mathbb{R}. \tag{15}$$

Put $h(x,t) = \lambda f(x,t) + \mu g_K(x,t)$. Using (12) and (15) we deduce that

$$|h(x,t)| \le \lambda C_f |t|^{p-1} + \mu C_g \text{ for a.e. } x \in \Omega, \text{ for any } t \in \mathbb{R}. \tag{16}$$

Now we develop a Moser iteration argument in the spirit of [**20**] to deduce the boundedness of u.

For any $T > 0$ and $\nu > 0$, we consider $\gamma(u) = \gamma_{T,\nu}(u) = u u_T^\nu \in W_0^{s,p}(\Omega)$, where $u_T = \min\{|u|, T\}$. Let us observe that

$$\psi'(a-b)(\gamma(a) - \gamma(b)) \ge |\Gamma(a) - \Gamma(b)|^p \text{ for any } a, b \in \mathbb{R}, \tag{17}$$

where $\psi(t) = \frac{|t|^p}{p}$ and $\Gamma(t) = \int_0^t (\gamma'(\tau))^{\frac{1}{p}} d\tau$.

Indeed, since γ is an increasing function, we can see that

$$(a-b)(\gamma(a) - \gamma(b)) \ge 0 \text{ for any } a, b \in \mathbb{R}.$$

Fix $a, b \in \mathbb{R}$ and we suppose that $a > b$. Then, from the definition of Γ and the

Jensen inequality we get

$$\begin{aligned}\psi'(a-b)(\gamma(a)-\gamma(b)) &= (a-b)^{p-1}(\gamma(a)-\gamma(b)) \\ &= (a-b)^{p-1}\int_b^a \gamma'(t)dt \\ &= (a-b)^{p-1}\int_b^a (\Gamma'(t))^p dt \\ &\geq \left(\int_b^a (\Gamma'(t))dt\right)^p = (\Gamma(a)-\Gamma(b))^p.\end{aligned}$$

In similar fashion we can prove that the above inequality is true for any $a \leq b$, so (17) holds. Taking $\gamma(u) = uu_T^{\nu}$ as test function in (P_K) and using (17) we can see that

$$\begin{aligned}&\|\Gamma(u)\|^p_{W_0^{s,p}(\Omega)} \\ &\leq \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}((uu_T^{\nu})(x)-(uu_T^{\nu})(y))\,dxdy \\ &= \int_\Omega [\lambda f(x,u)+\mu g_K(x,u)]uu_T^{\nu}dx. \end{aligned} \tag{18}$$

Since

$$\Gamma(u) \geq \frac{p}{p+\nu} uu_T^{\frac{\nu}{p}},$$

from the Sobolev inequality we can deduce that

$$\|\Gamma(u)\|^p_{W_0^{s,p}(\Omega)} \geq S_* \|\Gamma(u)\|^p_{L^{\bar{p}_s^*}(\Omega)} \geq \left(\frac{p}{p+\nu}\right)^p S_* \|uu_T^{\frac{\nu}{p}}\|^p_{L^{\bar{p}_s^*}(\Omega)},$$

where

$$S_* = \inf_{u \in W_0^{s,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{\|u\|^p_{L^{\bar{p}_s^*}(\Omega)}}.$$

This together with (16), (18) and the fact that $u_T \leq |u|$, implies that

$$\begin{aligned}\left(\frac{p}{p+\nu}\right)^p S_*^{-1} \|u u_T^{\frac{\nu}{p}}\|^p_{L^{\bar{p}_s^*}(\Omega)} &\leq \int_\Omega [\lambda f(x,u) + \mu g_K(x,u)] u u_T^\nu dx \\ &\leq \int_\Omega C_f \lambda |u|^{p+\nu} + \mu C_g |u|^{\nu+1} dx \\ &\leq (C_f\lambda + \mu C_g \max\{1, |\Omega|^{p-\frac{1}{p}}\})(1 + \|u\|^{p+\nu}_{L^{p+\nu}(\Omega)}).\end{aligned}$$

Then, if we suppose that $u \in L^{p+\nu}(\Omega)$, from Fatou's Lemma we can pass to the limit as $T \to \infty$ to get

$$\|u\|^{p+\nu}_{L^{\frac{\bar{p}_s^*}{p}(p+\nu)}(\Omega)} \leq \left(\frac{p+\nu}{p}\right)^p S_*(C_f\lambda + \mu C_g \max\{1, |\Omega|^{p-\frac{1}{p}}\})(1 + \|u\|^{p+\nu}_{L^{p+\nu}(\Omega)}). \tag{19}$$

Clearly, if $u \in L^{p+\nu}(\Omega)$, then we can infer that $u \in L^{\frac{\bar{p}_s^*}{p}(p+\nu)}(\Omega)$. Now we define the following sequence by induction:

$$\begin{cases} \nu_0 = \bar{p}_s^* - p \\ \nu_{j+1} = \dfrac{\bar{p}_s^*(p+\nu_j)}{p} - p. \end{cases}$$

It is easy to prove that the following relations hold for any $j \geq 0$:

(*i*) $\dfrac{\bar{p}_s^*}{p}(p+\nu_j) = \nu_{j+1} + p;$

(*ii*) $\nu_j \geq \left(\dfrac{\bar{p}_s^*}{p}\right)^j \nu_0$ and $\nu_j \to \infty$ as $j \to \infty$;

(*iii*) $\nu_j \leq (j+1)\left(\dfrac{\bar{p}_s^*}{p}\right)^j \nu_0;$

(*iv*) $\left(\dfrac{\bar{p}_s^*}{p} - 1\right)\nu_j = \left[\left(\dfrac{\bar{p}_s^*}{p}\right)^{j+1} - 1\right]\nu_0.$

In view of (19) and the property (*i*) we obtain

$$\begin{aligned}\|u\|^{p+\nu_{j-1}}_{L^{p+\nu_j}(\Omega)} &\leq \left(\frac{p+\nu_{j-1}}{p}\right)^p S_*\big(C_f\lambda + \mu C_g \max\{1, |\Omega|^{p-\frac{1}{p}}\}\big)\big(1 + \|u\|^{p+\nu_{j-1}}_{L^{p+\nu_{j-1}}(\Omega)}\big) \\ &\leq M_0 \left(\frac{p+\nu_{j-1}}{p}\right)^p \big(1 + \|u\|^{p+\nu_{j-1}}_{L^{p+\nu_{j-1}}(\Omega)}\big),\end{aligned} \tag{20}$$

where

$$M_0 := \max\{1, S_*(\lambda C_f + \mu C_g \max\{1, |\Omega|^{1-\frac{1}{p}}\})\}. \tag{21}$$

Let us denote by

$$\tau_j := \max\{1, \|u\|^{p+\nu_j}_{L^{p+\nu_j}(\Omega)}\}.$$

Taking into account (20), the definition of ν_j and property (iii) we get for any $j \geq 1$

$$\begin{aligned}
\log \tau_j &\leq \frac{p+\nu_j}{p+\nu_{j-1}}\left[p \log\left(M_0^{\frac{1}{p}}\left(\frac{p+\nu_{j-1}}{p}\right)\right) + \log(\tau_{j-1})\right] \\
&= \frac{\bar{p}_s^*}{p}\left[p \log\left(M_0^{\frac{1}{p}}\left(\frac{p+\nu_{j-1}}{p}\right)\right) + \log(\tau_{j-1})\right] \\
&\leq p\sum_{i=1}^{j}\left(\frac{\bar{p}_s^*}{p}\right)^i \log\left(M_0^{\frac{1}{p}}\left(\frac{p+\nu_{j-i}}{p}\right)\right) + \left(\frac{\bar{p}_s^*}{p}\right)^j \log \tau_0 \\
&= p\sum_{i=1}^{j}\left(\frac{\bar{p}_s^*}{p}\right)^i \log\left(M_0^{\frac{1}{p}}\left(1+\frac{\nu_{j-i}}{p}\right)\right) + \left(\frac{\bar{p}_s^*}{p}\right)^j \log \tau_0 \\
&\leq p\sum_{i=1}^{j}\left(\frac{\bar{p}_s^*}{p}\right)^i \log\left[M_0^{\frac{1}{p}}\left(1+\frac{\nu_0}{p}(j-i+1)\left(\frac{\bar{p}_s^*}{p}\right)^{j-i}\right)\right] + \left(\frac{\bar{p}_s^*}{p}\right)^j \log \tau_0. \qquad (22)
\end{aligned}$$

Now we set

$$\sigma := \frac{\bar{p}_s^*}{p} > 1.$$

Using (22) and (iv) we have

$$\begin{aligned}
&\log \max\{1, \|u\|_{L^{p+\nu_j}(\Omega)}\} = \frac{\log \tau_j}{p+\nu_j} = \frac{(\sigma-1)\log \tau_j}{(\sigma-1)p + (\sigma^{j+1}-1)\nu_0} \\
&\leq \frac{(\sigma-1)p}{(\sigma-1)p + (\sigma^{j+1}-1)\nu_0}\sum_{i=1}^{j}\sigma^i \log\left[M_0^{\frac{1}{p}}\left(1+\frac{\nu_0}{p}(j-i+1)\sigma^{j-i}\right)\right] \\
&+ \frac{(\sigma-1)\sigma^j \log \tau_0}{(\sigma-1)p + (\sigma^{j+1}-1)\nu_0}.
\end{aligned}$$

Since

$$\frac{(\sigma-1)}{(\sigma-1)p + (\sigma^{j+1}-1)\nu_0} < \frac{\sigma^{-j}}{\nu_0},$$

we can deduce that

$$
\begin{aligned}
&\log \max\{1, \|u\|_{L^{p+\nu_j}(\Omega)}\} \\
&\le \frac{p}{\nu_0} \sum_{i=1}^{j} \sigma^{-j+i} \log \left[M_0^{\frac{1}{p}} \left(1 + \frac{\nu_0}{p}(j-i+1)\sigma^{j-i} \right) \right] + \frac{(\sigma-1)\sigma^j \log \tau_0}{(\sigma-1)p + (\sigma^{j+1}-1)\nu_0} \\
&= \frac{p}{\nu_0} \sum_{k=0}^{j-1} \sigma^{-k} \log \left[M_0^{\frac{1}{p}} \left(1 + \frac{\nu_0}{p}(k+1)\sigma^{k} \right) \right] + \frac{(1-\sigma^{-1}) \log \tau_0}{(\sigma^{-j}-\sigma^{-j-1})p + (1-\sigma^{-j-1})\nu_0} \\
&=: (I) + (II).
\end{aligned}
$$

Using $\sigma > 1$ and $\log(1+t) \le t$ for any $t \ge 0$, we obtain

$$
\begin{aligned}
(I) &\le \frac{p}{\nu_0} \sum_{k=0}^{j-1} \sigma^{-k} \left[\log M_0^{\frac{1}{p}} + \log \left(\left(1 + \frac{\nu_0}{p}(k+1) \right) \sigma^k \right) \right] \\
&\le \frac{\log M_0}{\nu_0} \sum_{k=0}^{j-1} \sigma^{-k} + \sum_{k=0}^{j-1} \sigma^{-k}(k+1) + \frac{p}{\nu_0} \log \sigma \sum_{k=0}^{j-1} \sigma^{-k} k \\
&= \left(\frac{\log M_0}{\nu_0} + 1 \right) \sum_{k=0}^{j-1} \sigma^{-k} + \left(\frac{p}{\nu_0} \log \sigma + 1 \right) \sum_{k=0}^{j-1} \sigma^{-k} k. \qquad (23)
\end{aligned}
$$

On the other hand, from the definition of ν_0 we can see that

$$
(II) = \frac{(1-\sigma^{-1})\bar{p}_s^* \log \max\{1, \|u\|_{L^{\bar{p}_s^*}(\Omega)}\}}{(\sigma^{-j}-\sigma^{-j-1})p + (1-\sigma^{-j-1})\nu_0}. \qquad (24)
$$

Therefore, putting together (23) and (24) we have

$$
\begin{aligned}
&\log \max\{1, \|u\|_{L^{p+\nu_j}(\Omega)}\} \\
&\le \left(\frac{\log M_0}{\nu_0} + 1 \right) \sum_{k=0}^{j-1} \sigma^{-k} + \left(\frac{p}{\nu_0} \log \sigma + 1 \right) \sum_{k=0}^{j-1} \sigma^{-k} k \qquad (25) \\
&+ \frac{(1-\sigma^{-1})\bar{p}_s^* \log \max\{1, \|u\|_{L^{\bar{p}_s^*}(\Omega)}\}}{(\sigma^{-j}-\sigma^{-j-1})p + (1-\sigma^{-j-1})\nu_0}.
\end{aligned}
$$

Let us observe that

$$\begin{aligned}
&\lim_{j\to\infty}\left(\frac{\log M_0}{\nu_0}+1\right)\sum_{k=0}^{j-1}\sigma^{-k}+\left(\frac{p}{\nu_0}\log\sigma+1\right)\sum_{k=0}^{j-1}\sigma^{-k}k\\
&\quad+\frac{(1-\sigma^{-1})\bar{p}_s^*\log\max\{1,\|u\|_{L^{\bar{p}_s^*}(\Omega)}\}}{(\sigma^{-j}-\sigma^{-j-1})p+(1-\sigma^{-j-1})\nu_0}\\
&\quad=\left(\frac{\log M_0}{\nu_0}+1\right)\frac{1}{1-\sigma^{-1}}+\left(\frac{p}{\nu_0}\log\sigma+1\right)\sum_{k=0}^{+\infty}\sigma^{-k}k\\
&\quad+\log\max\{1,\|u\|_{L^{\bar{p}_s^*}(\Omega)}\}.
\end{aligned}\tag{26}$$

Then, in view of (25) and (26), and recalling that $\nu_j\to\infty$ as $j\to\infty$ (see (ii)), we deduce

$$\|u\|_{L^\infty(\Omega)}\le e^{M_1}\max\{1,\|u\|_{L^{\bar{p}_s^*}(\Omega)}\},\tag{27}$$

where

$$M_1:=\left(\frac{\log M_0}{\nu_0}+1\right)\frac{1}{1-\sigma^{-1}}+\left(\frac{p}{\nu_0}\log\sigma+1\right)\sum_{k=0}^{+\infty}\sigma^{-k}k.\tag{28}$$

Setting $K_1:=e^{M_1}$, and using the definition of S_* and $\|u\|_{W_0^{s,p}(\Omega)}<\rho$, we can see that (27) yields

$$\|u\|_{L^\infty(\Omega)}\le K_1\max\{1,S_*^{\frac{1}{p}}\rho\}.\tag{29}$$

Now we choose K in order to prove that $K_1\max\{1,S_*^{\frac{1}{p}}\rho\}\le K$. Let $K=K(\lambda,f,\rho)>0$ be such that

$$\begin{aligned}
&\log\left(\frac{K}{\max\{1,S_*^{\frac{1}{p}}\rho\}}\right)\\
&>\left[\left(\frac{\log\max\{1,S_*\lambda C_f\}}{\nu_0}+1\right)\frac{1}{1-\sigma^{-1}}+\left(\frac{p}{\nu_0}\log\sigma+1\right)\sum_{j=0}^{\infty}j\sigma^{-j}\right],
\end{aligned}$$

and take $\mu^* > 0$ such that for every $\mu \in [0, \mu^*]$

$$\log\left(\frac{K}{\max\{1, S_*^{\frac{1}{p}}\rho\}}\right)$$
$$> \left[\left(\frac{\log\max\{1, S_*(\lambda C_f + \mu C_g \max\{1, |\Omega|^{p-\frac{1}{p}}\})\}}{\nu_0} + 1\right)\frac{1}{1-\sigma^{-1}}\right.$$
$$\left. + \left(\frac{p}{\nu_0}\log\sigma + 1\right)\sum_{j=0}^{\infty} j\sigma^{-j}\right].$$

Set $\delta = \min\{\delta_0, \mu^*\}$. Then, for any $\mu \in [0, \delta]$, and recalling the definitions of K and μ^*, (21) and (28), we can infer that (29) yields

$$\|u\|_{L^\infty(\Omega)} \leq K_1 \max\{1, S_*^{\frac{1}{p}}\rho\} \leq K.$$

This shows that (14) holds true and we can end the proof of Theorem 1.1.

In conclusion, we present a direct application of our main result.

Example. Let $p = 2$ and let us consider the following functions

$$f(t) = \begin{cases} \min\{t^2, \sqrt{t}\} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0, \end{cases}$$

and $g(t) = e^t$. Then it is clear that $f, g \in L^\infty_{loc}(\mathbb{R}) \cap C(\mathbb{R})$, $f, g \geq 0$ and it holds

$$\lim_{|t|\to 0} \frac{f(t)}{|t|} = 0 = \lim_{|t|\to\infty} \frac{f(t)}{|t|}.$$

Hence $f \in \mathcal{A}$ satisfies assumptions $(f1)$-$(f3)$, and $g \in \mathcal{A}$. Now, fix $x_0 \in \Omega$ and let us choose $\tau > 0$ such that

$$\overline{B}(x_0, \tau) = \{x \in \mathbb{R}^N : |x - x_0| \leq \tau\} \subset \Omega.$$

Since

$$F(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{2}{3}t^{\frac{3}{2}} & \text{if } 0 < t \leq 1 \\ \frac{1}{3} + \frac{t^3}{3} & \text{if } t > 1, \end{cases}$$

there exists $t_0 > 0$ such that $F(t_0) > 0$. Take $\sigma_0 \in (0, 1)$ such that

$$2\sigma_0^N - 1 > 0.$$

Let us introduce the following auxiliary function

$$w^{t_0}_{\sigma_0}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus B(x_0,\tau) \\ \dfrac{t_0}{(1-\sigma)\tau}(\tau - |x-x_0|) & \text{if } x \in B(x_0,\tau) \setminus B(x_0,\sigma_0\tau) \\ t_0 & \text{if } x \in B(x_0,\sigma_0\tau). \end{cases}$$

Then, it is easy to verify (see pag. 18-20 in [**22**]) that $w^{t_0}_{\sigma_0} \in W^{s,2}_0(\Omega)$ and that there exist $\kappa_1, \kappa_2 > 0$ such that

$$\left(\iint_{\mathbb{R}^{2N}} \frac{|w^{t_0}_{\sigma_0}(x) - w^{t_0}_{\sigma}(y)|^2}{|x-y|^{N+2s}}\, dxdy \right)^{\frac{1}{2}} < \frac{t_0}{(1-\sigma_0)} \sqrt{\omega_N \tau^{N-2}(1-\sigma_0^N)\kappa_1\kappa_2}\,, \tag{30}$$

where $\omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$ is the volume of the unitary ball in $\mathbb{R}^N$.

Observing that $\max_{x\in\overline{\Omega}} |w^{t_0}_{\sigma_0}(x)| \leq t_0$, we can deduce that

$$\begin{aligned} J(w^{t_0}_{\sigma_0}) &= \int_\Omega F(w^{t_0}_{\sigma}(x))dx \\ &= \int_{B(x_0,\sigma_0\tau)} F(t_0)\, dx \;+ \int_{B(x_0,\tau)\setminus B(x_0,\sigma_0\tau)} F(w^{t_0}_{\sigma_0}(x))dx \\ &\geq \left(F(t_0)\sigma_0^N - (1-\sigma_0^N) \max_{|t|\leq t_0} F(t) \right) \omega_N \tau^N = F(t_0)(2\sigma_0^N - 1)\omega_N\tau^N. \qquad (31) \end{aligned}$$

Recalling the definitions of J, Φ and θ (see statement of Theorem 1.1), we can see that (30) and (31) yield the following upper bound for θ

$$0 < \theta < \frac{t_0^2(1-\sigma_0^N)\kappa_1\kappa_2}{2(1-\sigma_0)^2\tau^2 F(t_0)(2\sigma_0^N - 1)} := \theta_0.$$

Therefore, applying Theorem 1.1, we can deduce that for each compact interval $[a,b] \subset (\theta_0,\infty)$, there exists $\rho > 0$ such that, for every $\lambda \in [a,b]$ there exists $\delta > 0$ such that for any $\mu \in [0,\delta]$, the following problem

$$\begin{cases} (-\Delta)^s u = \lambda f(u) + \mu g(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

admits at least three weak solutions whose $W^{s,2}_0(\Omega)$-norms are less than ρ.

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