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On a semi-symmetric metric connection on the tangent bundle with the complete lift metric

Abstract. In this paper, we define a semi-symmetric metric connection on the tangent bundle equipped with complete lift metric. We compute the curvature tensors of this connection and study their properties. Also we investigate conditions for the tangent bundle to be locally conformally flat with respect to this connection.

Keywords. Semi-symmetric metric connection, tangent bundle, complete lift metric.

Mathematics Subject Classification (2010): 53B20, 53C07, 53C35.

1 - Introduction

Friedmann and Schouten [**1**] introduced the notion of a semi-symmetric linear connection on a differentiable manifold. Using Hayden's idea [**2**] of a metric connection with torsion, Yano [**6**] defined and investigated a semi-symmetric metric connection on a Riemannian manifold. He proved that a Riemannian manifold endowed with the semi-symmetric metric connection has vanishing curvature tensor if and only if the Riemannian manifold is conformally flat. Later, the generalization of this result for vanishing Ricci tensor of the semi-symmetric metric connection was shown by Imai in [**3**,**4**]. In this paper, first we shall define a semi-symmetric metric connection on the tangent bundle equipped with complete lift metric over a pseudo-Riemannian manifold. Secondly, we find all kinds of curvature tensors and study some properties of them. Finally we study conditions for the tangent bundle to be locally conformally flat with respect to the semi-symmetric metric connection.

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2 - Preliminaries

Let M be an $n-$dimensional differentiable manifold of class C^∞. The set of disjoint union $\underset{P\in M}{\cup}$ of the tangent spaces for all $P \in M$:

$$TM = \underset{P\in M}{\cup} T_P M$$

is called a tangent bundle over M. The natural projection π

$$\begin{aligned} \pi \quad &: \quad TM \to M \\ \widetilde{P} \quad &\to \quad \pi(\widetilde{P}) = P \end{aligned}$$

determines the correspondence of $(\widetilde{P} \to P)$ for any point $P \in M$. The set

$$\pi^{-1}(P) = \ \widetilde{P} \in T_P M$$

is called fibre on $P \in M$. Coordinate systems in M are denoted by (U, x^h), where U is the coordinate neighborhood and (x^h), $h = 1, ..., n$ are the coordinate functions. Let $(y^h) = (x^{\overline{h}})$, $\overline{h} = n{+}1, ..., 2n$ be the Cartesian coordinates in each tangent space $T_P M$ at $P \in M$ with respect to natural basis $\{\frac{\partial}{\partial x^h} \mid_P\}$, where P is an arbitrary point in U with local coordinates (x^h). Then we can introduce local coordinates (x^h, y^h) on the open set $\pi^{-1}(U) \subset TM$. Here the coordinate system of $(x^h, y^h) = (x^h, x^{\overline{h}})$ is called induced coordinates on $\pi^{-1}(U)$ from (U, x^h). The natural projection π is represented by $(x^h, y^h) \to (x^h)$. In the paper, we use Einstein's convention on repeated indices.

Let $X = X^h \frac{\partial}{\partial x^h}$ be the local expression in U of a vector field X on M. Let ∇ be a (torsion-free) linear connection on M. The vertical lift ${}^V X$ and the horizontal lift ${}^H X$ of X are given respectively by

$${}^V X = X^h \partial_{\overline{h}},$$

and

$${}^H X = X^h \partial_h - y^s \Gamma^h_{sk} X^k \partial_{\overline{h}}$$

with respect to the induced coordinates, where $\partial_h = \frac{\partial}{\partial x^h}, \partial_{\overline{h}} = \frac{\partial}{\partial y^h}$ and Γ^h_{jk} are the coefficients of the connection ∇.

With the connection ∇, we can introduce on each induced coordinate neighbourhood $\pi^{-1}(U)$ of TM a frame field which consists of the following $2n$ linearly independent vector fields:

$$\begin{aligned} E_j &= \partial_j - y^s \Gamma^h_{sj} \partial_{\overline{h}}, \\ E_{\overline{j}} &= \partial_{\overline{j}}. \end{aligned}$$

We are calling it as the adapted frame and, following [**7**], it will be written by $\{E_\beta\} = \left\{E_j, E_{\bar{j}}\right\}$. With respect to adapted frame $\{E_\beta\}$, the vertical lift ${}^V X$ and the horizontal lift ${}^H X$ of X is expressed by

$$\begin{aligned} {}^H X &= X^j E_j, \\ {}^V X &= X^j E_{\bar{j}}. \end{aligned}$$

3 - Semi-symmetric metric connection on the tangent bundle with the complete lift metric

3.1 - *Semi-symmetric connection*

Given a linear connection ∇ on an $n-$ dimensional differentiable manifold, if there exist a $1-$form ϕ, for which

$$T(X,Y) = \phi(Y)X - \phi(X)Y$$

then the connection ∇ is said to be semi-symmetric [**1**]. Here T denotes the torsion tensor of ∇, that is,

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

for all vector fields X, Y on M. On a (pseudo-)Riemannian manifold (M, g), a linear connection ∇ is called a metric connection [**2**] if

$$\nabla g = 0.$$

A linear connection ∇ is said to be a semi-symmetric metric connection if it is both semi-symmetric and metric [**6**].

Let M be an $n-$dimensional pseudo-Riemannian manifold with a pseudo-Riemannian metric g and let TM be its tangent bundle. The complete lift metric ${}^C g$ on TM is defined as follows:

$$\begin{aligned} {}^C g\left({}^H X, {}^H Y\right) &= 0, \\ {}^C g\left({}^H X, {}^V Y\right) &= {}^C g\left({}^V X, {}^H Y\right) = g\left(X, Y\right), \\ {}^C g\left({}^V X, {}^V Y\right) &= 0 \end{aligned}$$

for all vector fields X and Y on M, [**7**]. ${}^C g$ is a pseudo-Riemannian metric on TM. Note that the complete lift metric is also called the metric II. The covariant and contravariant components of the complete lift metric ${}^C g$ on TM are respectively given in the adapted local frame by

$${}^C g_{\alpha\beta} = \begin{pmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{pmatrix}$$

and

$$^{C}g^{\alpha\beta} = \begin{pmatrix} 0 & g^{ij} \\ g^{ij} & 0 \end{pmatrix}.$$

For the Levi-Civita connection $^{C}\nabla$ of the complete lift metric ^{C}g, we have:

Proposition 1. *The Levi-Civita connection* $^{C}\nabla$ *of* $(TM,^{C}g)$ *is given by*

$$\begin{cases} ^{C}\nabla_{E_i}E_j = \Gamma_{ij}^{k}E_k + y^s R_{sij}{}^{k}E_{\overline{k}}, \\ ^{C}\nabla_{E_i}E_{\overline{j}} = \Gamma_{ij}^{k}E_{\overline{k}}, \\ ^{C}\nabla_{E_{\overline{i}}}E_j = 0, \ ^{C}\nabla_{E_{\overline{i}}}E_{\overline{j}} = 0, \end{cases} \tag{3.1}$$

with respect to the adapted frame $\{E_\beta\}$, *where* Γ_{ij}^{h} *and* $R_{sij}{}^{k}$ *respectively denote components of the Levi-Civita connection* ∇ *and the Riemannian curvature tensor field* R *of the pseudo-Riemannian metric* g *on* M (*see* [**7**]).

Now we are interested in a semi-symmetric metric connection $\overline{\nabla}$ on $(TM,^{C}g)$. We denote the components of the semi-symmetric metric connection $\overline{\nabla}$ by $\overline{\Gamma}$. A semi-symmetric metric connection $\overline{\nabla}$ satisfies

$$\overline{\nabla}_\alpha(^{C}g_{\beta\gamma}) = 0 \text{ and } \overline{\Gamma}_{\alpha\beta}^{\gamma} - \overline{\Gamma}_{\beta\alpha}^{\gamma} - [E_\alpha, E_\beta] = \overline{T}_{\alpha\beta}^{\gamma}, \tag{3.2}$$

where $\overline{T}_{\alpha\beta}^{\gamma}$ are the components of the torsion tensor field of $\overline{\nabla}$. When the equation (3.2) is solved with respect to $\overline{\Gamma}_{\alpha\beta}^{\gamma}$, we find the following solution, [**2**]:

$$\overline{\Gamma}_{\alpha\beta}^{\gamma} = {}^{C}\Gamma_{\alpha\beta}^{\gamma} + U_{\alpha\beta}^{\gamma}, \tag{3.3}$$

where $^{C}\Gamma_{\alpha\beta}^{\gamma}$ are the components of the Levi-Civita connection of the complete lift metric ^{C}g,

$$U_{\alpha\beta\gamma} = \frac{1}{2}(\overline{T}_{\alpha\beta\gamma} + \overline{T}_{\gamma\alpha\beta} + \overline{T}_{\gamma\beta\alpha}) \tag{3.4}$$

and

$$U_{\alpha\beta\gamma} = U_{\alpha\beta}^{\epsilon}\,{}^{C}g_{\epsilon\gamma}, \ \overline{T}_{\alpha\beta\gamma} = T_{\alpha\beta}^{\epsilon}\,{}^{C}g_{\epsilon\gamma}.$$

We put

$$\overline{T}_{ij}{}^{\overline{k}} = y_j\delta_i^k - y_i\delta_j^k \tag{3.5}$$

and all other $\overline{T}_{\alpha\beta}^{\gamma}$ not related to $\overline{T}_{ij}{}^{\overline{k}}$ are assumed to be zero, where $y_i = y^s g_{si}$. By using (3.4) and (3.5), we get the only non-zero component of $U_{\alpha\beta}^{\gamma}$ as follows:

$$U_{ij}{}^{\overline{h}} = y_j\delta_i^h - y^h g_{ij}$$

with respect to the adapted frame. From (3.3) and (3.1), we have components of the semi-symmetric metric connection $\overline{\nabla}$ with respect to the complete lift metric ${}^{C}g$ as follows:

$$
\begin{array}{ll}
(i)\ \overline{\Gamma}_{ij}{}^{k} = \Gamma_{ij}{}^{k} & (v)\ \overline{\Gamma}_{ij}{}^{\overline{k}} = y^{s}R_{sij}{}^{k} + y_{j}\delta_{i}^{k} - y^{k}g_{ij} \\
(ii)\ \overline{\Gamma}_{\overline{i}\,j}{}^{k} = 0 & (vi)\ \overline{\Gamma}_{\overline{i}\,j}{}^{\overline{k}} = 0 \\
(iii)\ \overline{\Gamma}_{i\,\overline{j}}{}^{k} = 0 & (vii)\ \overline{\Gamma}_{i\,\overline{j}}{}^{\overline{k}} = \Gamma_{ij}{}^{k} \\
(iv)\ \overline{\Gamma}_{\overline{i}\,\overline{j}}{}^{k} = 0 & (viii)\ \overline{\Gamma}_{\overline{i}\,\overline{j}}{}^{\overline{k}} = 0.
\end{array}
$$

Hence we get:

Proposition 2. *The semi-symmetric metric connection $\overline{\nabla}$ of $(TM,{}^{C}g)$ is given by*

$$
\begin{cases}
\overline{\nabla}_{E_i}E_j = \Gamma_{ij}^{k}E_k + \{y^{s}R_{sij}{}^{k} + y_j\delta_i^k - y^k g_{ij}\}E_{\overline{k}},\\
\overline{\nabla}_{E_i}E_{\overline{j}} = \Gamma_{ij}^{k}E_{\overline{k}},\\
\overline{\nabla}_{E_{\overline{i}}}E_j = 0,\ \overline{\nabla}_{E_{\overline{i}}}E_{\overline{j}} = 0,
\end{cases}
$$

with respect to the adapted frame $\{E_\beta\}$, where Γ_{ij}^{h} and $R_{hji}{}^{s}$ respectively denote components of the Levi-Civita connection ∇ and the Riemannian curvature tensor field R of the pseudo-Riemannian metric g on M.

Given a pseudo-Riemannian metric g on a differentiable manifold M, another well known classical pseudo-Riemannian metric on TM is the metric $I + II$ defined by

$$
\begin{aligned}
\widetilde{g}\left(X^{H}, Y^{H}\right) &= g\left(X, Y\right)\\
\widetilde{g}\left(X^{H}, Y^{V}\right) &= \widetilde{g}\left(X^{V}, Y^{H}\right) = g\left(X, Y\right)\\
\widetilde{g}\left(X^{V}, Y^{V}\right) &= 0
\end{aligned}
$$

for all vector fields X, Y on M [**7**]. The metric $I + II$ has the components

$$
\widetilde{g}_{\alpha\beta} = \begin{pmatrix} g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}
$$

with respect to the adapted frame. Now, we consider the covariant derivation of the metric $I + II$ with respect to the semi-symmetric metric connection $\overline{\nabla}$.

One checks that

$$\begin{aligned}
\overline{\nabla}_k \overline{g}_{ij} &= E_k \widetilde{g}_{ij} - \overline{\Gamma}^h_{ki} \widetilde{g}_{hj} - \overline{\Gamma}^{\overline{h}}_{ki} \widetilde{g}_{\overline{h}j} - \overline{\Gamma}^h_{kj} \widetilde{g}_{ih} - \overline{\Gamma}^{\overline{h}}_{kj} \widetilde{g}_{i\overline{h}} \\
&= (\partial_k - y^s \Gamma^h_{sk} \partial_{\overline{k}}) g_{ij} - \Gamma^h_{ki} g_{hj} - (y^s R_{ski}{}^h + y_i \delta^h_k \\
&\quad - y^h g_{ki}) g_{hj} - \Gamma^h_{kj} g_{ih} - (y^s R_{skj}{}^h + y_j \delta^h_k - y^h g_{kj}) g_{ih} \\
&= \partial_k g_{ij} - \Gamma^h_{ki} g_{hj} - y^s R_{skij} - y_i g_{kj} + y_j g_{ki} - \Gamma^h_{kj} g_{ih} \\
&\quad - y^s R_{skji} - y_j g_{ik} + y_i g_{kj} \\
&= \partial_k g_{ij} - \Gamma^h_{ki} g_{hj} - \Gamma^h_{kj} g_{ih} \\
&= \nabla_k g_{ij} = 0,
\end{aligned}$$

$$\begin{aligned}
\overline{\nabla}_k \widetilde{g}_{i\overline{j}} &= E_k \widetilde{g}_{i\overline{j}} - \overline{\Gamma}^h_{ki} \overline{g}_{h\overline{j}} - \overline{\Gamma}^{\overline{h}}_{ki} \underbrace{\widetilde{g}_{\overline{h}\overline{j}}}_{0} - \underbrace{\overline{\Gamma}^h_{k\overline{j}}}_{0} \widetilde{g}_{ih} - \overline{\Gamma}^{\overline{h}}_{k\overline{j}} \widetilde{g}_{i\overline{h}} \\
&= \partial_k g_{ij} - \Gamma^h_{ki} g_{hj} - \Gamma^h_{kj} g_{ih} = \nabla_k g_{ij} = 0,
\end{aligned}$$

$$\begin{aligned}
\overline{\nabla}_k \overline{g}_{\overline{i}j} &= E_k \overline{g}_{\overline{i}j} - \underbrace{\overline{\Gamma}^h_{k\overline{i}}}_{0} \overline{g}_{hj} - \overline{\Gamma}^{\overline{h}}_{k\overline{i}} \overline{g}_{\overline{h}j} - \overline{\Gamma}^h_{kj} \overline{g}_{\overline{i}h} - \overline{\Gamma}^{\overline{h}}_{kj} \underbrace{\overline{g}_{\overline{i}\overline{h}}}_{0} \\
&= \partial_k g_{ij} - \Gamma^h_{ki} g_{hj} - \overline{\Gamma}^h_{kj} g_{ih} = \nabla_k g_{ij} = 0,
\end{aligned}$$

all others are automatically zero. Hence we can state following result.

Proposition 3. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^C g$ or the metric $I + II$. The semi-symmetric metric connection $\overline{\nabla}$ with respect to the complete lift metric ${}^C g$ is also a semi-symmetric metric connection with respect to the metric $I + II$.*

3.2 - *Curvature tensors*

The curvature tensor $\overline{R}$ of the semi-symmetric metric connection $\overline{\nabla}$ of $(TM, {}^C g)$ is obtained from the well-known formula

$$\overline{R}\left(\widetilde{X}, \widetilde{Y}\right) \widetilde{Z} = \overline{\nabla}_{\widetilde{X}} \overline{\nabla}_{\widetilde{Y}} \widetilde{Z} - \overline{\nabla}_{\widetilde{Y}} \overline{\nabla}_{\widetilde{X}} \widetilde{Z} - \overline{\nabla}_{[\widetilde{X}, \widetilde{Y}]} \widetilde{Z}$$

for all vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ on TM. From Proposition 2, we get the following.

Proposition 4. *The curvature tensor* $\overline{R}$ *of the semi-symmetric metric connection* $\overline{\nabla}$ *of* $(TM,{}^{C}g)$ *is given as follows:*

$$\begin{aligned}
\overline{R}(E_i,E_j)E_k &= R_{ijk}{}^{l}E_l+\{y^s\nabla_s R_{ijk}{}^{l}\}E_{\overline{l}},\\
\overline{R}(E_i,E_j)E_{\overline{k}} &= R_{ijk}{}^{l}E_{\overline{l}},\\
\overline{R}(E_i,E_{\overline{j}})E_k &= \{R_{ijk}{}^{l}+g_{ik}\delta_j^l-g_{jk}\delta_i^l\}E_{\overline{l}},\\
\overline{R}(E_{\overline{i}},E_j)E_k &= \{R_{ijk}{}^{l}+g_{ik}\delta_j^l-g_{jk}\delta_i^l\}E_{\overline{l}},\\
\overline{R}(E_{\overline{i}},E_{\overline{j}})E_k &= 0,\ \overline{R}(E_{\overline{i}},E_j)E_{\overline{k}}=0,\\
\overline{R}(E_i,E_{\overline{j}})E_{\overline{k}} &= 0,\ \overline{R}(E_{\overline{i}},E_{\overline{j}})E_{\overline{k}}=0
\end{aligned}$$

with respect to the adapted frame $\{E_\beta\}$.

Since the the Levi-Civita connection is the same for the complete lift metric ${}^{C}g$ and the metric $I+II$, their Riemannian curvature tensors coincide [**7**]. The Riemannian curvature tensor $\widetilde{R}$ of the Levi-Civita connection of the complete lift metric ${}^{C}g$ (or the metric $I+II$) is given by

$$\begin{aligned}
\widetilde{R}(E_i,E_j)E_k &= R_{ijk}{}^{l}E_l+\{y^s\nabla_s R_{ijk}{}^{l}\}E_{\overline{l}},\\
\widetilde{R}(E_i,E_j)E_{\overline{k}} &= R_{ijk}{}^{l}E_{\overline{l}},\\
\widetilde{R}(E_i,E_{\overline{j}})E_k &= R_{ijk}{}^{l}E_{\overline{l}},\\
\widetilde{R}(E_{\overline{i}},E_j)E_k &= R_{ijk}{}^{l}E_{\overline{l}},\\
\widetilde{R}(E_{\overline{i}},E_{\overline{j}})E_k &= 0,\ \widetilde{R}(E_{\overline{i}},E_j)E_{\overline{k}}=0,\\
\widetilde{R}(E_i,E_{\overline{j}})E_{\overline{k}} &= 0,\ \widetilde{R}(E_{\overline{i}},E_{\overline{j}})E_{\overline{k}}=0.
\end{aligned}$$

On comparing the curvature tensors of the Levi-Civita connection of the complete lift metric ${}^{C}g$ (or the metric $I+II$) and the semi-symmetric metric connection, we have the result below.

Corollary 1. *Let* (M,g) *be a pseudo-Riemannian manifold and* TM *be its tangent bundle equipped with the complete lift metric* ${}^{C}g$. *The curvature tensors of the Levi-Civita connection of the complete lift metric* ${}^{C}g$ *(or the metric* $I+II$ *) and the semi-symmetric metric connection coincide if and only if* $g_{ik}\delta_j^l-g_{jk}\delta_i^l=0$.

Theorem 1. *Let* (M,g) *be a pseudo-Riemannian manifold and* TM *be its tangent bundle equipped with the complete lift metric* ${}^{C}g$. *The curvature* $(0,4)-$*tensor* $\overline{R}$ *of the semi-symmetric metric connection* $\overline{\nabla}$ *holds the followings*

i) $\overline{R}_{\alpha\beta\gamma\sigma}+\overline{R}_{\beta\alpha\gamma\sigma}=0,$

ii) $\overline{R}_{\alpha\beta\gamma\sigma} + \overline{R}_{\alpha\beta\sigma\gamma} = 0$,

iii) $\overline{R}_{\alpha\beta\gamma\sigma} - \overline{R}_{\gamma\sigma\alpha\beta} = 0$ *if and only if* $g_{ik}g_{jh} - g_{jk}g_{ih} = 0$.

Proof. On lowering the upper index of the curvature tensor $\overline{R}$ of the semi-symmetric metric connection $\overline{\nabla}$, the non-zero components of the curvature $(0,4)-$tensor are obtained as follows:

$$\begin{aligned} \overline{R}_{ijkh} &= y^s \nabla_s R_{ijkh}, \\ \overline{R}_{ijk\,\overline{h}} &= R_{ijkh}, \\ \overline{R}_{ij\,\overline{k}h} &= R_{ijkh}, \\ \overline{R}_{i\,\overline{j}\,kh} &= R_{ijkh} + g_{ik}g_{jh} - g_{jk}g_{ih}, \\ \overline{R}_{\overline{i}\,j\,kh} &= R_{ijkh} + g_{ik}g_{jh} - g_{jk}g_{ih}. \end{aligned} \tag{3.6}$$

i) and *ii)* The results immediately follows from above relations.

iii) With the help of above relations, when we calculate the $(0,4)-$tensor: $\overline{P}_{\alpha\beta\gamma\sigma} = \overline{R}_{\alpha\beta\gamma\sigma} - \overline{R}_{\gamma\sigma\alpha\beta}$, it follows that

$$\begin{aligned} \overline{P}_{ijk\overline{h}} &= g_{ik}g_{jh} - g_{jk}g_{ih}, \\ \overline{P}_{\overline{i}\,jkh} &= g_{ik}g_{jh} - g_{jk}g_{ih}, \\ \overline{P}_{ij\,\overline{k}\,h} &= g_{ik}g_{jh} - g_{jk}g_{ih}, \\ \overline{P}_{i\,\overline{j}\,kh} &= g_{ik}g_{jh} - g_{jk}g_{ih} \end{aligned}$$

all others are zero. The last relations imply that $\overline{P}_{\alpha\beta\gamma\sigma} = 0$ if and only if $g_{ik}g_{jh} - g_{jk}g_{ih} = 0$. □

Let $\overline{K}_{\alpha\beta} = \overline{R}_{\sigma\alpha\beta}{}^{\sigma}$ denotes the Ricci tensor of the semi-symmetric metric connection $\overline{\nabla}$. Then

$$\begin{aligned} \overline{K}_{jk} &= 2K_{jk} + (1-n)g_{jk}, \\ \overline{K}_{j\overline{k}} &= 0, \\ \overline{K}_{\overline{j}\,k} &= 0, \\ \overline{K}_{\overline{j}\,\overline{k}} &= 0 \end{aligned} \tag{3.7}$$

from which the following result follows:

Theorem 2. *Let (M,g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^{C}g$. The Ricci tensor of the semi-symmetric metric connection $\overline{\nabla}$ is symmetric.*

A (pseudo-)Riemannian manifold (M, g) is called an Einstein manifold if the relation

$$K_{jk} = \lambda g_{jk}$$

holds with a scalar function λ, where K is the Ricci tensor of (M, g). It immediately follows from (3.7) that:

Theorem 3. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^C g$. Then TM is Ricci flat with respect to the semi-symmetric metric connection $\overline{\nabla}$ if and only if M is a Einstein manifold.*

A (pseudo-)Riemannian manifold (M, g) is called Ricci semi-symmetric if the following condition is satisfied: [**5**]

$$R(X, Y).K = 0,$$

where $R(X, Y)$ is a linear operator acting as a derivation on the Ricci curvature tensor K of (M, g).

The curvature operator $\overline{R}(\widetilde{X}, \widetilde{Y})$ is a differential operator on TM for all vector fields $\widetilde{X}$ and $\widetilde{Y}$. Now, we operate the curvature operator $\overline{R}(\widetilde{X}, \widetilde{Y})$ to the Ricci curvature tensor $\overline{K}$, that is, for all $\widetilde{Z}, \widetilde{W}$, we consider the condition $(\overline{R}(\widetilde{X}, \widetilde{Y})\overline{K})(\widetilde{Z}, \widetilde{W}) = 0$. In the case, we shall call TM Ricci semi-symmetric with respect to the semi-symmetric metric connection $\overline{\nabla}$.

In the adapted frame $\{E_\beta\}$, the tensor $(\overline{R}(\widetilde{X}, \widetilde{Y})\overline{K})(\widetilde{Z}, \widetilde{W})$ is locally expressed as follows:

$$(\overline{R}(\widetilde{X}, \widetilde{Y})\overline{K})(\widetilde{Z}, \widetilde{W})_{\alpha\beta\gamma\theta} = \overline{R}_{\alpha\beta\gamma}{}^{\varepsilon}\overline{K}_{\varepsilon\theta} + \overline{R}_{\alpha\beta\theta}{}^{\varepsilon}\overline{K}_{\gamma\varepsilon}. \tag{3.8}$$

Similarly, in local coordinates,

$$((R(X, Y)K)(Z, W)_{ijkl} = R_{ijk}{}^{p}K_{pl} + R_{ijl}{}^{p}K_{kp}.$$

Theorem 4. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^C g$. Then TM is Ricci semi-symmetric with respect to the semi-symmetric metric connection $\overline{\nabla}$ if and only if M is Ricci semi-symmetric.*

Proof. By putting $\alpha = i, \beta = j, \gamma = k, \theta = l$ in (3.8), we find

$$\begin{aligned} & (\overline{R}(\widetilde{X},\widetilde{Y})\overline{K})(\widetilde{Z},\widetilde{W})_{ijkl} \\ = & \ \overline{R}_{ijk}{}^{h}\overline{K}_{hl} + \overline{R}_{ijl}{}^{h}\overline{K}_{kh} \\ = & \ R_{ijk}^{h}[2K_{hl} + (1-n)g_{hl}] + R_{ijl}^{h}[2K_{kh} + (1-n)g_{kh}] \\ = & \ 2(R_{ijk}^{h}K_{hl} + R_{ijl}^{h}K_{kh}) + (1-n)\underbrace{(R_{ijkl} + R_{ijlk})}_{0} \\ = & \ 2(R_{ijk}{}^{h}R_{hl} + R_{ijl}{}^{h}R_{kh}) \\ = & \ 2((R(X,Y)Ric)(Z,W))_{ijkl} \end{aligned}$$

all the others being zero. This finishes the proof. □

For the scalar curvature $\overline{r}$ of the semi-symmetric metric connection $\overline{\nabla}$ with respect to ${}^{C}g$, we find

$$\overline{r} = \overline{K}_{\alpha\beta}{}^{C}g^{\alpha\beta} = \overline{K}_{jk}{}^{C}g^{jk} + \overline{K}_{\bar{j}k}{}^{C}g^{\bar{j}k} + \overline{K}_{j\bar{k}}{}^{C}g^{j\bar{k}} + \overline{K}_{\bar{j}\bar{k}}{}^{C}g^{\bar{j}\bar{k}} = 0.$$

Then we can state the following.

Theorem 5. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^{C}g$. The scalar curvature of TM with the semi-symmetric metric connection $\overline{\nabla}$ with respect to ${}^{C}g$ vanishes.*

3.3 - *Locally conformally flatness*

In this section we investigate locally conformally flatness property of $(TM, {}^{C}g)$ with respect to the semi-symmetric metric connection $\overline{\nabla}$.

Theorem 6. *Let (M, g) be a pseudo-Riemannian manifold and TM be its tangent bundle equipped with the complete lift metric ${}^{C}g$. Then TM is locally conformally flat with respect to the semi-symmetric metric connection $\overline{\nabla}$ if and only if M is locally flat and $g_{ik}g_{jh} - g_{jk}g_{ih} = 0$.*

Proof. Here, we prove the only necessary conditions of Theorem because the sufficient condition directly follows. Let $\overline{\nabla}$ be the semi-symmetric metric connection on the tangent bundle $(TM, {}^{C}g)$. The tangent bundle $(TM, {}^{C}g)$ is locally conformally flat with respect to the semi-symmetric metric connection $\overline{\nabla}$ if and only if the components of the curvature $(0,4)-$tensor $\overline{R}$ of TM satisfy the following relation:

$$\begin{aligned}\overline{R}_{\alpha\gamma\beta\mu} &= -\frac{\overline{r}}{2(2n-1)(n-1)}\left\{{}^{C}g_{\alpha\beta}{}^{C}g_{\gamma\mu} - {}^{C}g_{\alpha\mu}{}^{C}g_{\gamma\beta}\right\} \\ &\quad + \frac{1}{2(n-1)}\left({}^{C}g_{\gamma\mu}\overline{K}_{\alpha\beta} - {}^{C}g_{\alpha\mu}\overline{K}_{\gamma\beta} + {}^{C}g_{\alpha\beta}\overline{K}_{\gamma\mu} - {}^{C}g_{\gamma\beta}\overline{K}_{\alpha\mu}\right).\end{aligned}$$

From (3.7) we find

$$\overline{R}_{ijkh} = 0$$

$$\overline{R}_{i\bar{j}kh} = \frac{1}{(n-1)}\left(g_{jh}R_{ik} - g_{jk}R_{ih}\right) - \frac{1}{2}\left(g_{ik}g_{jh} - g_{jk}g_{ih}\right) \tag{3.9}$$

$$\overline{R}_{\bar{i}jkh} = \frac{1}{(n-1)}\left(g_{ik}R_{jh} - g_{ih}R_{jk}\right) - \frac{1}{2}\left(g_{ik}g_{jh} - g_{jk}g_{ih}\right) \tag{3.10}$$

$$\overline{R}_{ij\bar{k}h} = \frac{1}{(n-1)}\left(g_{ik}R_{jh} - g_{jk}R_{ih}\right) - \frac{1}{2}\left(g_{ik}g_{jh} - g_{jk}g_{ih}\right) \tag{3.11}$$

$$\overline{R}_{ijk\bar{h}} = \frac{1}{(n-1)}\left(g_{jh}R_{ik} - g_{ih}R_{jk}\right) - \frac{1}{2}\left(g_{jh}g_{ik} - g_{ih}g_{jk}\right). \tag{3.12}$$

Subtracting (3.12) (resp.(3.11)) from (3.9) (resp. (3.10)), we respectively have

$$g_{ik}g_{jh} - g_{jk}g_{ih} = \frac{1}{(n-1)}\left(g_{ih}R_{jk} - g_{jk}R_{ih}\right)$$

$$g_{ik}g_{jh} - g_{jk}g_{ih} = \frac{1}{(n-1)}\left(g_{jk}R_{ih} - g_{ih}R_{jk}\right)$$

from which, the sum of the last two equations gives

$$g_{ik}g_{jh} - g_{jk}g_{ih} = 0.$$

In this case, by means of (3.10) and (3.7), we get $R_{ijkl} = 0$ which completes the proof. □

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