Riv. Mat. Univ. Parma, Vol. **9** (2018), 227-254

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Framing the bases of Impulsive Mechanics of constrained systems into a jet–bundle geometric context

Abstract. We illustrate how the different kinds of constraints acting on an impulsive mechanical system can be described in the geometric setup given by the configuration space–time bundle $\pi_t : \mathcal{M} \to \mathbb{E}$ and its first jet extension $\pi : J_1(\mathcal{M}) \to \mathcal{M}$ in a way that ensures total compliance with coordinate and frame invariance requirements of Classical Mechanics. We specify the differences between geometric and constitutive characterizations of a constraint. We point out the relevance of the role played by the concept of frame of reference, underlining when the frame independence is mandatorily required and when a choice of a frame is an inescapable need. The thorough rationalization allows the introduction of unusual but meaningful kinds of constraints, such as unilateral kinetic constraints or breakable constraints, and of new theoretical aspects, such as the possible dependence of the impulsive reaction by the active forces acting on the system.

Keywords. Fibred space–time, impulsive constraint, constitutive characterization.

Mathematics Subject Classification (2010): 70F35, 70–02, 70G45.

Introduction

The theory of jet–extensions of fibred manifolds (see, e.g. [**Sau89**,**Pom78**]) is, at the present time, the best common ground where both the invariant description provided by a differential geometric setup and the techniques provided by Mathematical Analysis about time dependent ordinary differential equations can be synergically applied to study Classical Mechanics.

Received: July 11, 2018; accepted in revised form: November 16, 2018.

Unfortunately this synergy is not satisfactorily effective for the study of Classical Impulsive Mechanics. Due to presence of the impulsive aspects, with their intrinsic discontinuity of the velocities, the study of Classical Impulsive Mechanics presents two alternative approaches: the analytical one, powered by all the techniques of Mathematical Analysis such as measure differential equations, bounded values and absolutely continuous functions and the related techniques but weakened by the difficulties in having a coordinate and/or frame invariant character, and the geometric one that, thanks to the powerful techniques of generic coordinates and vectors fields, satisfies the invariance requirements of a mechanical theory but that can menage the equations of motion with great difficulty.

A remarkable exception to this forced choice consists in the so–called "*event–driven*" algebraic approach to Impulsive Mechanics. In fact it gives rise to finite, and not differential, evolution equations, so that, in this context, the importance of the techniques of Mathematical Analysis is partly downsized, and the geometric setup gains greater prominence.

We adopt the geometric environment of the first order jet–bundle $J_1(\mathcal{M})$ of the configuration space–time bundle $\pi_t : \mathcal{M} \to \mathbb{E}$ and its subbundles, as described in [**MP91**, **MP97**, **Pas08**, **Pas18a**]. It fits to the requirements of allowing a natural coordinate and frame invariant description, of embodying the "first order" aspect of the theory (that is, results pertaining position and velocity of the system) and, when we restrict our attention to the event–driven algebraic approach, of giving powerful techniques to model and analyze the impulsive phenomenon of both free and constrained systems.

This paper has the main aim of rationalizing and gathering in a single work the majority of known ideas about the jet–bundle approach to the basic aspects of Impulsive Mechanics, with a detailed analysis of the various types of impulsive constraints acting on the system. Along the process of rationalization, we also spot and underline both the consistency of the description with coordinate and frame invariance properties and crudities, inaccuracies, or even the inconsistencies of the common geometric frameworks given by a configuration space $\mathcal{Q}$ or its product bundles $\mathbb{R} \times \mathcal{Q}$ and $\mathbb{E} \times \mathcal{Q}$.

The very brief Preliminaries section contains the non–formal and heuristic descriptions of the impulsive mechanical problem and of the range of applicability of the impulsive approach to the evolution of a mechanical system.

Section 1 has an introductory character: we briefly recall the geometric setup of the configuration space–time bundle $\mathcal{M} \to \mathbb{E}$ and the absolute velocity space–time bundle $J_1(\mathcal{M}) \to \mathcal{M}$. We show that this geometric context forms a very natural environment where describing impulsive behaviors, and for formulating the Integrated, or Impulsive, Newton Law (briefly INL), as presented e.g. in [**LCA22**, **Par65**, **Per53**, **Str00**], in the correct causal formulation.

Section 2 concerns the geometric setup for a time–dependent and frame–independent description of impulsive mechanical systems subject to constraints. We show that we can embody the wide variety of constraints possibly acting on the system in a single geometric structure formed by suitable subbundles of the geometric setup describing a free system. Using this structure, we can classify the impulsive characters of constraints on the basis of their geometric properties: positional or kinetic, bilateral or unilateral, permanent or instantaneous, single or multiple.

Section 3 concerns the constitutive possibilities of the various kinds of constraints, classified depending on the nature of their action on the impulsive system. We distinguish the constraints as ideal or non–ideal, with or without friction, breakable or unbreakable.

Section 4 concludes the paper presenting a condensed summary of rationalizations, ideas and clarifications distributed along the paper, specifying the few having an innovative character.

The list of possible references about impulsive constrained systems is very huge, and a bibliography claiming to be exhaustive on the argument should be excessively long compared to the length of the paper. Moreover, only few works would be reasonably pertinent to the specific approach presented in the paper. Therefore, the list of references has been based on the minimality criterion of making the paper self–consistent.

Preliminaries

An impulsive behavior in a "single point" of a mechanical system with a finite number of degrees of freedom is a time–evolution of the system such that the map assigning the position of the system is continuous for every instant while the map assigning the velocity of the system is continuous in all except a single instant, and in this instant of discontinuity the velocities are subject to a finite jump.

This manifestly heuristic definition will be clearer once the correct geometric setup of the problem will be described. It is clear that no real mechanical system has such a behavior, and that as a matter of principle the correct description of such phenomenon should involve a "very small" but not singular time interval in which the evolution should be studied using deformation, elasticity, thermodynamics, acoustics and so on. Then the definition is a purely theoretical model, a limit situation of motions of systems where some changes of velocity are sudden enough to be considered instantaneous, or at least such that we are not interested in a detailed knowledge (or we are not able to perform a precise analysis) of what happens to the system between two "very near" instants. However, overlooking this evident formal vulnerability, the examples of evolutions of mechanical systems for which such an approximation provides a useful context of analysis are numerous and physically meaningful.

The choice of considering a single point of discontinuity of the velocity does not weaken the approach. Every time evolution of a mechanical system can be studied locally (with respect to time) and every impulsive behavior of the system can happen only in an isolated instant, forerun and followed by two non singular time intervals where the maps of position and velocity are both continuous. This justifies the so called event–driven approach, where the regular motion of the system just before the instant of discontinuity determines the initial conditions of the impulsive problem, whose "solution" determines the initial conditions of the regular motion of the system just after the discontinuity.

1 - Free Systems

In this section, also in order to fix notation, we briefly describe the geometric setup suited to study Impulsive Mechanics of a free system, and we introduce the impulsive problem in the geometric context. The content is very standard, and can be easily found in the huge literature about fiber bundle techniques in non–relativistic field theory (see e.g. the books [**Sau89**, **Pom78**, **CP86**, **dLR90**], the works of C. Duval *et al.* [**DH09**, **BDP83**], the works of M. de Léon *et al.* [**VCdLdD05**, **dMd97**, **IDLea98**], the works of E. Massa *et al.* [**MP91**, **MP97**] and the references therein. For a broader and more focused presentation of the same arguments, see also [**Pas18b**, **Pas08**, **Pas18a**]).

1.1 - *Geometry of Free systems*

The *configuration space–time* of a mechanical system with a finite number n of degrees of freedom is a fiber bundle $\pi_t : \mathcal{M} \to \mathbb{E}$ where $\mathcal{M}$ is a $(n+1)$-dimensional differentiable manifold and $\mathbb{E}$ is the 1-dimensional Euclidean space. The fibers of the bundle $\mathcal{M}$ are diffeomorphic to an n-dimensional manifold $\mathcal{Q}$, usually called the *configuration space* of the system. $\mathcal{M}$ can be locally described by fibred coordinates (t, x^i), $i = 1, \ldots, n$.

The first jet–extension $\pi : J_1(\mathcal{M}) \to \mathcal{M}$ of the bundle $\mathcal{M}$ is the *absolute velocity space–time* of the system. It is the $(2n+1)$–dimensional affine subbundle of the tangent bundle $T(\mathcal{M})$ given by the vectors tangent to any possible motion $\gamma : \mathbb{E} \to \mathcal{M}$ of the system in any point. The elements of $J_1(\mathcal{M})$ have the form

$\mathbf{p} = \frac{\partial}{\partial t} + p^1 \frac{\partial}{\partial x^1} + \ldots + p^n \frac{\partial}{\partial x^n}$, and they are called *time–like vectors.* $J_1(\mathcal{M})$ can be locally described by jet–coordinates $(t, x^i, \dot{x}^i)$, $i = 1, \ldots, n$.

The affine jet–bundle $J_1(\mathcal{M})$ is modelled on the $(2n+1)$–dimensional vector bundle of the vertical vectors, that is the vectors that are tangent to the fiber of $\mathcal{M}$. The elements of $V(\mathcal{M})$ have the form $\mathbf{V} = V^1 \frac{\partial}{\partial x^1} + \ldots + V^n \frac{\partial}{\partial x^n}$, and they are called *space–like vectors.* $V(\mathcal{M})$ too can be locally described by the coordinates $(t, x^i, \dot{x}^i)$, $i = 1, \ldots, n$. The (fibred) action of $V(\mathcal{M})$ on $J_1(\mathcal{M})$ is the fibred sum

$$
\begin{array}{rccccl}
+: & J_1(\mathcal{M}) \times V(\mathcal{M}) & \to & J_1(\mathcal{M}) & \text{s.t.} \\
& (\mathbf{p}, \mathbf{V}) & \rightsquigarrow & \mathbf{p} + \mathbf{V} & \Leftrightarrow \\
& ((t, x^i, p^i), (t, x^i, V^i)) & \rightsquigarrow & (t, x^i, p^i + V^i). &
\end{array}
\tag{1}
$$

We endow $\mathcal{M}$ with a vertical positive definite metric, that is a space–like scalar product

$$
\begin{array}{rccccl}
\Phi: & V(\mathcal{M}) \times_{\mathcal{M}} V(\mathcal{M}) & \to & \mathbb{R} & \text{s.t.} \\
& (\mathbf{V}_1, \mathbf{V}_2) & \rightsquigarrow & \Phi(\mathbf{V}_1, \mathbf{V}_2) & \Leftrightarrow \\
& ((t, x^i, V_1^i), (t, x^i, V_2^i)) & \rightsquigarrow & g_{ij}\, V_1^i\, V_2^j. &
\end{array}
$$

The vertical metric usually takes into account the massive properties of the system and then the positive definite matrix g_{ij} is called the *mass matrix* of the system. Of course, since $g_{ij} = \Phi(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, the expression of g_{ij} depends on the coordinate system.

Remark 1. Since an aim of the paper is to illustrate a geometrical framework for the event–driven algebraic approach to Impulsive Mechanics, we have to deal only with velocities and impulses acting on the system, and we do not have to deal with accelerations and forces (with one exception in Section 3). Then we do not introduce in details the "acceleration space–time" $J_2(\mathcal{M})$ with its structures and properties. For the same reason, we do not need the concepts of time–derivative and connection on $\mathcal{M}$. About these arguments, see e.g. [**MP91**, **MP97**] and the references therein. ♢

A global frame of reference for $\mathcal{M}$ consists of a complete time–like vector field $\mathbf{h} : \mathcal{M} \to J_1(\mathcal{M})$ having the local expression $\mathbf{h} = \frac{\partial}{\partial t} + H^1(t, x^i) \frac{\partial}{\partial x^1} + \ldots + H^n(t, x^i) \frac{\partial}{\partial x^n}$. The integral lines of $\mathbf{h}$ gives a global one to one correspondence between the fibers $\{\pi_t^{-1}(t^*) \,|\, t^* \in \mathbb{E}\}$ of $\mathcal{M}$.
It is clearly possible to give a notion of local frame of reference, renouncing the completeness of the vector field $\mathbf{h}$. All the following arguments still hold in a local sense. The set of frames of reference will be denoted with $\mathcal{H}$.

Remark 2. The naive choice of $\mathbb{E} \times \mathcal{Q}$ (or, worst, $\mathbb{R} \times \mathcal{Q}$ or even $\mathbb{R} \times \mathbb{R}^n$) as space–time bundle for the system implies an intrinsic choice of a frame of reference: the invariant time–like vector field $\frac{\partial}{\partial t} \in T(\mathbb{E} \times \mathcal{Q})$. In this case, the (possibly unwitting) use of this intrinsically defined frame of reference undermines the frame independent description of the behavior of a mechanical system, even more so for impulsive systems. In fact, velocity, that is the basic concept in Impulsive Mechanics, is by its very nature a physical quantity referred to and depending on the knowledge of a frame of reference, and usual statements involving velocity, such as for example the preservation of kinetic energy in an impulsive phenomenon, could be meaningless if the assignment of the frame of reference where they are formulated is lacking and, on the other side, could be not frame invariant if assigned (possibly unwittingly) in a fixed frame. $\diamondsuit$

A frame $\mathbf{h}$ determines a diffeomorphism, usually called *relativization*, $\Delta_{\mathbf{h}} : J_1(\mathcal{M}) \to V(\mathcal{M})$ of the affine bundle $J_1(\mathcal{M})$ with the vector bundle $V(\mathcal{M})$ such that $\Delta_{\mathbf{h}}(\mathbf{p}) = \mathbf{p} - \mathbf{h}(\pi(\mathbf{p}))$. The space–like vector $\mathbf{V_h}(\mathbf{p}) = \Delta_{\mathbf{h}}(\mathbf{p}) = \mathbf{p} - \mathbf{h}(\pi(\mathbf{p}))$ is the *relative velocity* of the absolute velocity $\mathbf{p}$ *with respect to the frame* $\mathbf{h}$. The function

$$\mathcal{K}_{\mathbf{h}} : J_1(\mathcal{M}) \to \mathbb{R} \quad \text{s.t.} \quad \mathcal{K}_{\mathbf{h}}(\mathbf{p}) = \frac{1}{2}\Phi\left(\mathbf{V_h}(\mathbf{p}), \mathbf{V_h}(\mathbf{p})\right)$$

is the *kinetic energy* of the system *with respect to the frame* $\mathbf{h}$.

1.2 - *Impulsive problem for free systems*

The impulsive problem for free systems is easily framed in the geometric context described above. Roughly speaking, it consists in the determination of the so–called *right velocity* of the system once a *left velocity* of the system and an impulse acting on the system are known. The intrinsic character of the impulsive phenomenon, that is independent of the presence of a frame of reference, implies that the left and right–velocities are elements $\mathbf{p}_L, \mathbf{p}_R$ of $J_1(\mathcal{M})$. The impulse is an element $\mathbf{I} \in V(\mathcal{M})$ and the mechanical law governing the phenomenon is simply the action (1)

$$\begin{array}{ccccc} J_1(\mathcal{M}) \times V(\mathcal{M}) & \to & J_1(\mathcal{M}) & & \text{s.t.} \\ (\mathbf{p}_L, \mathbf{I}) & \rightsquigarrow & \mathbf{p}_R = \mathbf{p}_L + \mathbf{I}. & & \end{array} \tag{2}$$

Each frame of reference $\mathbf{h}$ determines a relativization of this action so that $(\mathbf{p}_R - \mathbf{h}) = (\mathbf{p}_L - \mathbf{h}) + \mathbf{I}$ or, that is the same, $(\mathbf{V_h})_R - (\mathbf{V_h})_L = \mathbf{I}$ that closely resembles the usual formulation of the Impulsive Newton Law (INL) and that also exhibits the independence of the jump of velocities by the frame of reference.

Moreover, the INL in the form (2) has the correct causal structure as equation of the impulsive motion of the free system. In fact the impulse, in this case called active impulse $\mathbf{I}_{act}$, can be thought of as a fibred map

$$\mathbf{I}_{act} : J_1(\mathcal{M}) \to \quad V(\mathcal{M}) \quad \text{s.t.} \quad \mathbf{p} \rightsquigarrow \mathbf{I}_{act}(\mathbf{p})$$

and then the INL assumes the causally correct form $\mathbf{p}_R = \mathbf{p}_L + \mathbf{I}_{act}(\mathbf{p}_L)$.

2 - Geometry of constrained systems

A constraint acting on a system is, in the widest sense, any limitation imposed on the possible motions of the system. The extreme generality of this statement is usually clarified by distinguishing different types of constraints: for instance, positional or kinetic, bilateral or unilateral, permanent or instantaneous, ideal or not, with or without friction, breakable or unbreakable. Only some of these distinctions are of geometric nature, and they are described in this section. Moreover, of course, only some combinations of these types of constraints have a clear physical meaning.

In order to make effectively applicable the general definition and the consequent classification, and recalling that the causal structure of the Newton's Second Law implies that constraints on a system can be assigned only as limitations on its admissible positions and/or velocities, we introduce additional geometric structures on the framework set up for free systems (see [**MP91**, **MP97**, **Pas06**, **IDLea98**, **Pas12**, **Pas08**]).

2.1 - *Geometry of positional constraints*

A positional constraint consists, roughly speaking, in a constraint on the admissible space–time configuration of the system. It can be modelled with a fibred subbundle $i : \mathcal{S} \to \mathcal{M}$ that at this stage we suppose of constant dimension $r + 1$ with $0 < r < n$, and without boundary. The bundle $\mathcal{S}$ determines the following additional geometric objects and structures relative to the system:

1) The affine subbundle $i_* : J_1(\mathcal{S}) \to J_1(\mathcal{M})$ of the absolute velocities that are tangent to $\mathcal{S}$ and the vector subbundle $i_* : V(\mathcal{S}) \to V(\mathcal{M})$ of the vertical vectors that are tangent to the fibers of $\mathcal{S}$. Both these bundles are locally described by coordinates $(t, q^1, \ldots, q^r, \dot{q}^1, \ldots, \dot{q}^r)$.

2) The pull–back bundle $\pi : (i_*)^*(J_1(\mathcal{M})) \to \mathcal{S}$, that is the *velocity space–time* of the system when the system is *in contact* with the constraint $\mathcal{S}$, and the pull–back bundle $\pi : (i_*)^*(V(\mathcal{M})) \to \mathcal{S}$, that is the bundle of

the space vectors of the system when the system is *in contact* with the constraint $\mathcal{S}$. Both these bundles are locally described by coordinates $(t, q^1, \ldots, q^r, \dot{x}^1, \ldots, \dot{x}^n)$.

3) Thanks to the vertical metric Φ, the splitting of the vector bundle $(i_*)^*(V(\mathcal{M})) = i_*(V(\mathcal{S})) \oplus V^{\perp}(\mathcal{S})$ with its associated projection operators $\mathcal{P}^{\|}, \mathcal{P}^{\perp}$.

4) The splitting of the affine bundle $(i_*)^*(J_1(\mathcal{M})) = i_*(J_1(\mathcal{S})) \oplus V^{\perp}(\mathcal{S})$, where in this case the direct sum $\oplus$ reflects the action (1), and its associated projection operators $\mathcal{P}_{\mathcal{S}}, \mathcal{P}_{\mathcal{S}}^{\perp}$.

5) The subclass $\mathcal{H}_{\mathcal{S}}$ of the frames of reference $\mathbf{h}_{\mathcal{S}}$ of $\mathcal{M}$ such that $(\mathbf{h}_{\mathcal{S}})_{\rfloor \mathcal{S}}$ is tangent to $\mathcal{S}$. The elements of $\mathcal{H}_{\mathcal{S}}$ are called the (possible) *rest frames* of $\mathcal{S}$. If $\mathcal{S}$ is assigned in cartesian form $f_\rho(t, x^1, \ldots, x^n) = 0$, they are characterized by the conditions $\mathbf{h}_{\mathcal{S}}(f_\rho) = 0 \,\forall \rho = 1, \ldots, n-r$.

Example 1. Let $\mathcal{M}$ be the configuration space–time of a massive particle freely moving in a 3–dimensional euclidean space, so that $\mathcal{M} = \mathbb{E} \times \mathbb{E}^3$. If $\mathcal{M}$ is described by cartesian coordinates (t, x, y, z), let $\mathcal{S}_1$ be the subbundle described by the immersion $i : \mathcal{S}_1 \to \mathcal{M}$ such that $(t, q^1, q^2) \rightsquigarrow (t, x = q^1, y = q^2, 0)$ or alternatively by the cartesian representation $z = 0$. △

Example 2. Let $\mathcal{M}$ be as above and let $\mathcal{S}_2$ be the subbundle described by the immersion $i : \mathcal{S}_2 \to \mathcal{M}$ such that $(t, q) \rightsquigarrow (t, x = q, 0, 0)$ or alternatively by the cartesian representation $y = z = 0$. △

2.1.1 - Bilateral positional constraint

If all the possible motions $\gamma : \mathbb{E} \to \mathcal{M}$ of the system obey the condition $\gamma : \mathbb{E} \to \mathcal{S} \subset \mathcal{M}$, then all the admissible configurations for the system belong to $\mathcal{S}$ and all the absolute velocities $\dot{\gamma} : \mathbb{E} \to J_1(\mathcal{M})$ belong to $J_1(\mathcal{S})$ and are then tangent to $\mathcal{S}$. In this case, $\mathcal{S}$ represents a bilateral positional permanent constraint. Such a kind of constraint, depending on the nature of the system (for instance, in absence of active impulses), could be absorbed in the construction itself of the space–time bundle $\mathcal{M}$. We will go back on this kind of constraints only when they will have an impulsive character (for instance when the system is subject to an active impulse).

2.1.2 - Unilateral positional constraint

The pull–back bundle $(i_*)^*(J_1(\mathcal{M}))$ is formed by the time–like vectors $\mathbf{p} \in J_1(\mathcal{M})$ such that $\pi(\mathbf{p}) \in i(\mathcal{S})$, but $\mathbf{p}$ is not necessarily tangent to $\mathcal{S}$: then $(i_*)^*(J_1(\mathcal{M}))$ is the natural geometric framework fit to analyze the behavior of $\mathcal{S}$ viewed as a unilateral positional constraint.

Definition 2.1. A positional constraint $\mathcal{S}$ is called unilateral in a point $s \in \mathcal{S}$ if two sets $\mathcal{L}_s(J_1(\mathcal{M})) \subset (i_*)^*_s(J_1(\mathcal{M}))$ and $\mathcal{R}_s(J_1(\mathcal{M})) \subset (i_*)^*_s(J_1(\mathcal{M}))$ are assigned so that the space $(i_*)^*_s(J_1(\mathcal{M}))$ of the time–like vector of $\mathcal{M}$ applied in s can be written as the disjoint union

$$(i_*)^*_s(J_1(\mathcal{M})) = \mathcal{L}_s(J_1(\mathcal{M})) \cup (i_*)_s(J_1(\mathcal{S})) \cup \mathcal{R}_s(J_1(\mathcal{M})) \tag{3}$$

The constraint $\mathcal{S}$ is called unilateral if it is unilateral in every point $s \in \mathcal{S}$.

The set $\mathcal{L}_s(J_1(\mathcal{M})) \cup (i_*)_s(J_1(\mathcal{S}))$ is the set of the *admissible left velocities* for the system in contact with $\mathcal{S}$ in the point $s \in \mathcal{S}$, while the set $\mathcal{R}_s(J_1(\mathcal{M})) \cup (i_*)_s(J_1(\mathcal{S}))$ is the set of the *admissible right velocities* for the system in contact with $\mathcal{S}$ in the point $s \in \mathcal{S}$. We define $\mathcal{L}_{\mathcal{S}}(J_1(\mathcal{M})) = \bigcup_{s\in\mathcal{S}} \mathcal{L}_s(J_1(\mathcal{M}))$, $\mathcal{R}_{\mathcal{S}}(J_1(\mathcal{M})) = \bigcup_{s\in\mathcal{S}} \mathcal{R}_s(J_1(\mathcal{M}))$. The set $\mathcal{L}_{\mathcal{S}}(J_1(\mathcal{M}))$ is also called the set of *incoming* or *entrance* velocities, while the set $\mathcal{R}_{\mathcal{S}}(J_1(\mathcal{M}))$ is that of *outgoing* or *exit* velocities.

An important result about the geometric structures determined by the subbundle $\mathcal{S}$ and the contact bundle $(i_*)^*(J_1(\mathcal{M}))$ is the following (see [**Pas05a**]):

Theorem 2.1. *Given a frame of reference* $\mathbf{h} \in \mathcal{H}_{\mathcal{S}}$, *the diagrams*

$$\begin{array}{ccc} (i_*)^*(J_1(\mathcal{M})) & \xrightarrow{\Delta_{\mathbf{h}}} & (i_*)^*(V(\mathcal{M})) \\ \downarrow{\scriptstyle \mathcal{P}^{\perp}_{\mathcal{S}}} & & \downarrow{\scriptstyle \mathcal{P}^{\perp}} \\ V^{\perp}(\mathcal{S}) & = & V^{\perp}(\mathcal{S}) \end{array} \qquad \begin{array}{ccc} (i_*)^*(J_1(\mathcal{M})) & \xrightarrow{\Delta_{\mathbf{h}}} & (i_*)^*(V(\mathcal{M})) \\ \downarrow{\scriptstyle \mathcal{P}_{\mathcal{S}}} & & \downarrow{\scriptstyle \mathcal{P}^{\parallel}} \\ i_*(J_1(\mathcal{S})) & \xrightarrow{\Delta_{\mathbf{h}}} & i_*(V(\mathcal{S})) \end{array}$$

commutes. The diagrams does not commute if $\mathbf{h} \notin \mathcal{H}_{\mathcal{S}}$.

The theorem clarifies the frame invariance properties of the orthogonal and parallel components of an absolute velocity with respect to a positional constraint.

Given an absolute velocity $\mathbf{p} \in (i_*)^*(J_1(\mathcal{M}))$ and a generic frame $\mathbf{h} \in \mathcal{H}$ we can construct the two space–like vectors $\mathbf{V}^{\perp}_1(\mathbf{p}) = \mathcal{P}^{\perp}_{\mathcal{S}}(\mathbf{p})$, and $\mathbf{V}^{\perp}_2(\mathbf{p}) =$

$\mathcal{P}^{\perp}(\mathbf{p}-\mathbf{h})$. Both are elements of $V^{\perp}(\mathcal{S})$: the first is manifestly independent of the frame $\mathbf{h}$ while the second in general depends on $\mathbf{h}$, so that in general $\mathbf{V}_1^{\perp}(\mathbf{p}) \neq \mathbf{V}_2^{\perp}(\mathbf{p})$. The theorem states that the two vectors coincide if and only if $\mathbf{h}$ is a rest frame for $\mathcal{S}$. Then the best definition of orthogonal component $\mathbf{V}^{\perp}(\mathbf{p})$ of the absolute velocity $\mathbf{p}$ with respect to $\mathcal{S}$ can be given only for frames in the class $\mathcal{H}_{\mathcal{S}}$, and it is $\mathbf{V}^{\perp}(\mathbf{p}) = \mathcal{P}_{\mathcal{S}}^{\perp}(\mathbf{p}) = \mathcal{P}^{\perp}(\mathbf{p}-\mathbf{h}_{\mathcal{S}})$.

In a similar way, given $\mathbf{p} \in (i_*)^*(J_1(\mathcal{M}))$ and $\mathbf{h} \in \mathcal{H}$ we can construct the two space–like vectors $\mathbf{V}_1^{\|}(\mathbf{p}) = \mathcal{P}_{\mathcal{S}}(\mathbf{p})-\mathbf{h}$ and $\mathbf{V}_2^{\|}(\mathbf{p}) = \mathcal{P}^{\|}(\mathbf{p}-\mathbf{h})$: the second is an element of $i_*(V(\mathcal{S}))$ for every $\mathbf{h} \in \mathcal{H}$, while the first in general is an element of $(i_*)^*(V(\mathcal{M}))$ and $\mathbf{V}_1^{\|}(\mathbf{p}) \in i_*(V(\mathcal{S}))$ if and only if $\mathbf{h} \in \mathcal{H}_{\mathcal{S}}$. Then once again in general $\mathbf{V}_1^{\|}(\mathbf{p}) \neq \mathbf{V}_2^{\|}(\mathbf{p})$. The theorem states that the two vectors coincide if and only if $\mathbf{h}$ is a rest frame for $\mathcal{S}$. Then once again the best definition of parallel component $\mathbf{V}_{\mathbf{h}}^{\|}(\mathbf{p})$ of the absolute velocity $\mathbf{p}$ with respect to $\mathcal{S}$ can be given only for frames in the class $\mathcal{H}_{\mathcal{S}}$, and it is $\mathbf{V}_{\mathbf{h}}^{\|}(\mathbf{p}) = \mathcal{P}_{\mathcal{S}}(\mathbf{p})-\mathbf{h}_{\mathcal{S}} = \mathcal{P}^{\|}(\mathbf{p}-\mathbf{h}_{\mathcal{S}})$.

Remark 3. Note that, even if we restrict the choice of frames in the class $\mathcal{H}_{\mathcal{S}}$, the parallel component depends on the frame $\mathbf{h}$. This will play a crucial role in the next section especially regarding the concept of friction. ◊

Example 3. If a massive particle moving in the 3–dimensional euclidean space impacts with a plane, the orthogonal component of the impact velocity is univocally determined by the geometry of the system (the plane and its class of rest frames), while the tangent component of the velocity is not univocally determined. In fact, naively speaking, the tangent component of the velocity changes if, in the contact point, the plane is at rest or if it is formed, for example, by a conveyor belt. △

If $\mathcal{S} \subset \mathcal{M}$ is of codimension 1, then $(i_*)^*(V^{\perp}(\mathcal{S}))$ has dimension 1 and we can choose a (possibly but not necessarily unit) vector $\mathbf{U}^{\perp}$ such that $V^{\perp}(\mathcal{S}) = Lin(\mathbf{U}^{\perp})$. Then, for every $\mathbf{p} \in (i_*)^*(J_1(\mathcal{M}))$, we can evaluate the sign of $\Phi(\mathbf{V}^{\perp}(\mathbf{p}), \mathbf{U}^{\perp})$: recalling that, of course, $\Phi(\mathbf{V}^{\perp}(\mathbf{p}), \mathbf{U}^{\perp}) = 0$ implies $\mathbf{V}^{\perp}(\mathbf{p}) = 0$ and so $\mathbf{p} \in i_*(J_1(\mathcal{S}))$, we can define, for example,

$$\begin{aligned} &\mathbf{p} \in \mathcal{L}_{\mathcal{S}}(J_1(\mathcal{M})) \Leftrightarrow \Phi(\mathbf{V}^{\perp}(\mathbf{p}), \mathbf{U}^{\perp}) < 0 \\ &\mathbf{p} \in \mathcal{R}_{\mathcal{S}}(J_1(\mathcal{M})) \Leftrightarrow \Phi(\mathbf{V}^{\perp}(\mathbf{p}), \mathbf{U}^{\perp}) > 0. \end{aligned} \tag{4}$$

Example 4. Going back to the previous Ex.1 and Ex.2, the constraint $\mathcal{S}_1$ is naturally unilateral because it is of codimension 1. We can set $\mathbf{U}^{\perp} = \frac{1}{\sqrt{m}}\frac{\partial}{\partial z}$ (or simply $\mathbf{U}^{\perp} = \frac{\partial}{\partial z}$) so that $V^{\perp}(\mathcal{S}_1) = Lin(\mathbf{U}^{\perp}) = Lin(\frac{\partial}{\partial z})$ and then the rule (4) determines the admissible left and right velocities for the particle when in contact with $\mathcal{S}_1$.

Differently, the constraint $\mathcal{S}_2$ is not naturally unilateral. Since, roughly speaking, the constraint consists in a 1–dimensional line in a 3–dimensional space, every non–tangent velocity of the particle in contact with $\mathcal{S}_2$ can be either an entrance or an exit velocity for the particle. Of course, we can choose, although in an arbitrary way, a splitting (3). $\triangle$

R e m a r k 4. An effective geometrization of unilateral constraints allows the construction of an effective geometric model of positional constraints with boundary. If we choose $\mathbf{U}^{\perp} = \frac{\partial}{\partial z}$ and $\mathcal{L}_{\mathcal{S}_1}(J_1(\mathcal{M})), \mathcal{R}_{\mathcal{S}_1}(J_1(\mathcal{M}))$ defined as in (4), the constraint $\mathcal{S}_1$ of Ex.1 is the geometric model of the positional constraint with boundary given by the condition $z \geq 0$. However, the constraint $\mathcal{S}_2$ together with an arbitrary assignment of $\mathcal{L}_{\mathcal{S}_2}(J_1(\mathcal{M}))$ and $\mathcal{R}_{\mathcal{S}_2}(J_1(\mathcal{M}))$, due to its codimension greater than 1, has not a clear counterpart in terms of positional constraint with boundary. $\diamondsuit$

2.1.3 - Multiple unilateral positional constraints

It is clear that the presence of two ore more bilateral positional constraints $\mathcal{S}_1, \mathcal{S}_2$ can be modelled with a single bilateral positional constraint given by their intersection $\mathcal{S}_1 \cap \mathcal{S}_2$. Then we restrict our attention to multiple unilateral positional constraints.

Def. 2.1 allows to highlight the difference between a “genuine” constraint of codimension greater than 1 and the simultaneous action of more than one constraint of codimension 1. Let us refer to Ex.1 and Ex.2.

E x a m p l e 5. Let $\mathcal{M}$ be the space–time bundle of Ex.1 and Ex.2, and, with a slight abuse of notation, let $\mathcal{S}_y, \mathcal{S}_z$ be the subbundles described respectively by the immersions $i_y : \mathcal{S}_y \to \mathcal{M}$ such that $(t,x,z) \rightsquigarrow (t,x,0,z)$, $i_z : \mathcal{S}_z \to \mathcal{M}$ such that $(t,x,y) \rightsquigarrow (t,x,y,0)$ or alternatively by the respective cartesian representations $\mathcal{S}_y = \{y = 0\}, \mathcal{S}_z = \{z = 0\}$. Then we have $\mathcal{S}_2 = \mathcal{S}_y \cap \mathcal{S}_z$, but, although $\mathcal{S}_2$ and $\mathcal{S}_y \cap \mathcal{S}_z$ are the same subbundle of $\mathcal{M}$, from the mechanical point of view they have different behaviors. In fact we already saw that the constraint $\mathcal{S}_2$ is not naturally unilateral since it does not admit a natural choice of entrance and exit velocities. On the contrary, the constraint $\mathcal{S}_y \cap \mathcal{S}_z$ allows a natural splitting (3): with obvious notation, we have $V^{\perp}(\mathcal{S}_y) = Lin(\frac{\partial}{\partial y}), V^{\perp}(\mathcal{S}_z) = Lin(\frac{\partial}{\partial z})$ and we can define

(5)

$$\mathbf{p} \in \mathcal{L}_{\mathcal{S}_y \cap \mathcal{S}_z}(J_1(\mathcal{M})) \quad \Leftrightarrow \quad \Phi(\mathbf{V}^{\perp}_{\mathcal{S}_y}(\mathbf{p}), \tfrac{\partial}{\partial y}) < 0 \qquad \text{or} \qquad \Phi(\mathbf{V}^{\perp}_{\mathcal{S}_z}(\mathbf{p}), \tfrac{\partial}{\partial z}) < 0$$

$$\mathbf{p} \in \mathcal{R}_{\mathcal{S}_y \cap \mathcal{S}_z}(J_1(\mathcal{M})) \quad \Leftrightarrow \quad \begin{cases} \Phi(\mathbf{V}^{\perp}_{\mathcal{S}_y}(\mathbf{p}), \tfrac{\partial}{\partial y}) \geq 0 \\ \\ \Phi(\mathbf{V}^{\perp}_{\mathcal{S}_z}(\mathbf{p}), \tfrac{\partial}{\partial z}) > 0 \end{cases} \quad \text{or} \quad \begin{cases} \Phi(\mathbf{V}^{\perp}_{\mathcal{S}_y}(\mathbf{p}), \tfrac{\partial}{\partial y}) > 0 \\ \\ \Phi(\mathbf{V}^{\perp}_{\mathcal{S}_z}(\mathbf{p}), \tfrac{\partial}{\partial z}) \geq 0. \end{cases}$$

Then the rule (5) determines the admissible left and right velocities for the particle when in contact with $\mathcal{S}_y \cap \mathcal{S}_z$.

The unilateral constraints $\mathcal{S}_y$ and $\mathcal{S}_z$, together with the condition (5), can be considered the model of the unilateral constraint with boundary given by $\begin{cases} y \geq 0 \\ z \geq 0 \end{cases}$. Instead, the constraint $\mathcal{S}_2$ does not give positional restrictions to the particle. △

Definition 2.2. A multiple unilateral positional constraint $\mathcal{S}$ is a regular intersection of unilateral positional constraints $\mathcal{S}_i, i = 1, \ldots, r \geq 2$ of codimension 1. The intersection is regular if the vectors $\{\mathbf{U}^{\perp}_{\mathcal{S}_i}, i = 1, \ldots, r\}$ are linearly independent in every point of $\mathcal{S} = \bigcap_{i=1,\ldots,r} \mathcal{S}_i$.

2.2 - *Geometry of kinetic constraints*

Kinetic constraints are, roughly speaking, those that fix limitations on the admissible velocities of the system without fixing limitations on its configurations. They are mainly divided into permanent and instantaneous kinetic constraints, having different geometric characteristics (see e.g. [**MP91**, **Pas06**]).

2.2.1 - `Permanent kinetic constraints`

A permanent kinetic constraint can be modelled with a fibred subbundle $i : \mathcal{A} \to J_1(\mathcal{M})$. If all the possible motions $\gamma : \mathbb{E} \to \mathcal{M}$ of the system obey the condition $j_1(\gamma) : \mathbb{E} \to \mathcal{A} \subset J_1(\mathcal{M})$, then all the possible absolute velocities of the system belong to $\mathcal{A}$. In this case, $\mathcal{A}$ represents a permanent kinetic constraint. Such a kind of constraint, once again depending on the nature of the system (for instance, in absence of active impulses), could have not an impulsive character. Once again, we will go back on this kind of constraints only when they will have an impulsive character.

The affine structure of the constraint $\mathcal{A}$ viewed as subbundle of $J_1(\mathcal{M})$ is a usual requirement in order to ensure that the impulsive problem holds on to be governed by an INL of type (1). In this case, thanks to the vertical metric Φ, $\mathcal{A}$ determines:

1) the splitting $V(\mathcal{M}) = V(\mathcal{A}) \oplus V^{\perp}(\mathcal{A})$ with its associated projection operators;

2) the splitting $J_1(\mathcal{M}) = \mathcal{A} \oplus V^{\perp}(\mathcal{A})$ with its associated projection operators;

3) the subclass $\mathcal{H}_{\mathcal{A}}$ of frames of $\mathcal{H}$ having image in $\mathcal{A}$ (that is, naively speaking, the class of the rest frames of $\mathcal{A}$).

Remark 5. It is possible to assign an impulsive problem for a mechanical system by assigning a kinetic constraint $\mathcal{A} \subset J_1(\mathcal{M})$ such that, for a fixed instant t_0 or for the points of an assigned subset $\mathcal{N} \subset \mathcal{M}$, $\mathcal{A} = \mathcal{R}_{\mathcal{N}}(J_1(\mathcal{M}))$ is the set of admissible right velocities (while $\mathcal{L}_{(\mathcal{M}\setminus\mathcal{N})}(J_1(\mathcal{M})) = J_1(\mathcal{M})$ itself). This is the well known case of the so–called *inert constraints* (see e.g. [**IDLea01**, **Pas05b**]). $\diamondsuit$

2.2.2 - Instantaneous kinetic constraints

An instantaneous kinetic constraint is a kinetic constraint that acts on the system only during the instant of the impulsive behavior, usually the instant of collision of the system with a positional constraint $\mathcal{S}$. An instantaneous kinetic constraint $\mathcal{B}$ is then modelled with a fibred subbundle $i : \mathcal{B} \to J_1(\mathcal{S})$.

Example 6. The pure rolling conditions in the contact point for a sphere of radius R moving in a 3–dimensional halfspace and impacting with a horizontal plane is an example of instantaneous kinetic constraint. Describing the space–time $\mathcal{M}$ with the usual coordinates $(t, x, y, z, \psi, \vartheta, \varphi)$ where (x, y, z) are the coordinates of the center of the sphere and $(\psi, \vartheta, \varphi)$ are the Euler angles, the positional constraint $\mathcal{S}$ is given by the condition $z - R = 0$, and the instantaneous kinetic constraint $\mathcal{B}$ is given by the equations

$$\begin{cases} \dot{x} - R\dot{\vartheta}\sin\psi + R\dot{\varphi}\sin\vartheta\cos\psi = 0 \\ \dot{y} + R\dot{\vartheta}\cos\psi + R\dot{\varphi}\sin\vartheta\sin\psi = 0. \end{cases} \tag{6}$$

$\triangle$

Once again the affine structure of the constraint $\mathcal{B}$ viewed as subbundle of $J_1(\mathcal{S})$ is a usual requirement for $\mathcal{B}$, so that the differences $\mathbf{B} = b_1 - b_2, b_i \in \mathcal{B}$ between elements of $\mathcal{B}$ form a vector subspace $V(\mathcal{B})$ of $V(\mathcal{S})$. In this case, the INL (1) gives once again the equation of motion of the system.

If $\mathcal{B}$ is an affine subbundle of $J_1(\mathcal{S})$, the vector subbundle $i_* : V(\mathcal{B}) \to V(\mathcal{S})$ determines

1) the splitting $V(\mathcal{S}) = V(\mathcal{B}) \oplus V^{\perp}(\mathcal{B})$ with its associated projection operators;

2) the splitting $J_1(\mathcal{S}) = \mathcal{B} \oplus V^{\perp}(\mathcal{B})$ with its associated projection operators;

3) the subclass $\mathcal{H}_{\mathcal{B}}$ of frames of $\mathcal{H}_{\mathcal{S}}$ having image in $\mathcal{B}$ (that is, naively speaking, the class of the rest frames of $\mathcal{B}$).

Taking into account the immersion $i : \mathcal{S} \to \mathcal{M}$, we obtain the splittings $V(\mathcal{M}) = V(\mathcal{B}) \oplus V^{\perp}(\mathcal{B}) \oplus V^{\perp}(\mathcal{S})$ and $J_1(\mathcal{M}) = \mathcal{B} \oplus V^{\perp}(\mathcal{B}) \oplus V^{\perp}(\mathcal{S})$. In this case, given $\mathbf{p} \in (i_*)^*(J_1(\mathcal{M}))$, we have that $\mathbf{p} = \mathcal{P}^{\parallel}_{\mathcal{B}}(\mathbf{p}) + \mathbf{V}^{\perp}_{\mathcal{B}}(\mathbf{p}) + \mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p})$. We already discussed the invariant properties of $\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p})$ with respect to the class $\mathcal{H}_{\mathcal{S}}$ of rest frames of $\mathcal{S}$. It is a straightforward corollary of Th. 2.1 that the space–like vector $\mathbf{V}^{\perp}_{\mathcal{B}}(\mathbf{p})$ has the same invariant properties with respect to the subclass $\mathcal{H}_{\mathcal{B}}$.

2.2.3 - Further remarks on kinetic constraints

Kinetic constraints are so naturally embodied in the geometric setup that their definition can be easily extended to unilateral cases: for instance, the kinetic conditions (6) expressing the pure rolling of a sphere on a horizontal plane can be modified in the form

$$\begin{cases} \dot{x} - R\dot{\vartheta}\sin\psi + R\dot{\varphi}\sin\vartheta\cos\psi \geq 0 \\ \dot{y} + R\dot{\vartheta}\cos\psi + R\dot{\varphi}\sin\vartheta\sin\psi \geq 0\,. \end{cases} \tag{7}$$

Of course being the conditions (7) mathematically correct, the physical meaning of such a constraint is hard to conceive. However, the following mechanical system gives an example of permanent kinetic unilateral constraint.

Example 7. A disk moves with its boundary in contact with a horizontal rough plane and with the axis of the disk kept in horizontal position. The system can be described by coordinates $(t, x, y, \vartheta, \varphi)$ where (x, y) are the coordinates of the center of the disk, ϑ is the orientation of the vertical plane of the disk with respect to a fixed vertical plane and φ is the orientation of the disk with respect to a horizontal plane containing its axis. The disk is subject to a coaster brake, so that the condition $\dot{\varphi} \geq 0$ holds. If moreover the disk is subject to the pure rolling kinetic constraint, $\mathcal{A}$ is expressed by the conditions

$$\begin{cases} \dot{x} + R\dot{\varphi}\cos\vartheta &= 0 \\ \dot{y} + R\dot{\varphi}\sin\vartheta &= 0 \\ \dot{\varphi} &\geq 0\,. \end{cases}$$

△

Remark 6. At odds with the naturalness of the formal definition of kinetic constraints, the effective assignment of the kinetic restrictions exerted on a mechanical system could require several details, possibly the knowledge of a frame of reference. The pure rolling conditions, classically expressed in the form "the difference between the velocities of the contact point of the system and of the contact point of the constraint is zero" (see, e.g. [**LCA22**]), since the second velocity cannot be viewed as an element of $J_1(\mathcal{M})$, requires the knowledge of a frame where the two velocities can be compared. ◊

2.3 - *Miscellaneous*

The simultaneous presence of constraints of the same kind acting on the system was already taken into account by introducing multiple constraints in the case of unilateral positional constraints and by the definition itself of (permanent or instantaneous) kinetic constraints. However, an impulsive system can be simultaneously subject to different kinds of constraints.

Example 8. In a classical "billiard situation", a sphere rolling on a horizontal plane impacts with a vertical wall, with a pure rolling condition of the sphere in the contact point of the vertical wall during the impact. This is a very natural example of system simultaneously subject to three different kinds of constraints: a unilateral positional constraint $\mathcal{S}$ (the vertical wall), a permanent kinetic constraint $\mathcal{A}$ (the pure rolling condition on the horizontal floor) and an instantaneous kinetic constraint $\mathcal{B}$ (the pure rolling condition in the contact point of the sphere with the vertical wall). △

Of course, the simultaneous presence of unilateral positional constraints $\mathcal{S}$ and kinetic constraints $\mathcal{A}, \mathcal{B}$ provides the geometric context with all the structures determined by each constraint and with the structures that can be constructed with them.

2.4 - *The global diagram*

The whole geometric construction fitting to frame the impulsive mechanical problem for a constrained system is then the following Diagram 1.

With a mild abuse of notation, identifying some bundles with their immersions, we can focus our attention on the central part of the diagram, synthetically represented by Diagram 2.

Later on, with the same mild abuse of notation used in the previous diagrams, every frame $\mathbf{h}$ having $\mathcal{S}, \mathcal{A}$ or $\mathcal{B}$ as index are intended such that, once restricted to the points of $\mathcal{S}$, the frame has image in $J_1(\mathcal{S}), \mathcal{A}$ or $\mathcal{B}$ respectively,

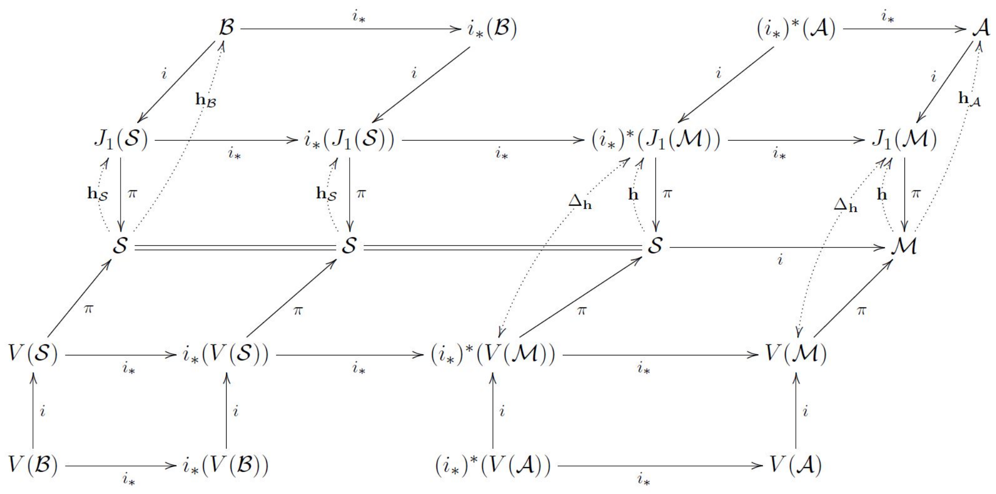

Diagram 1

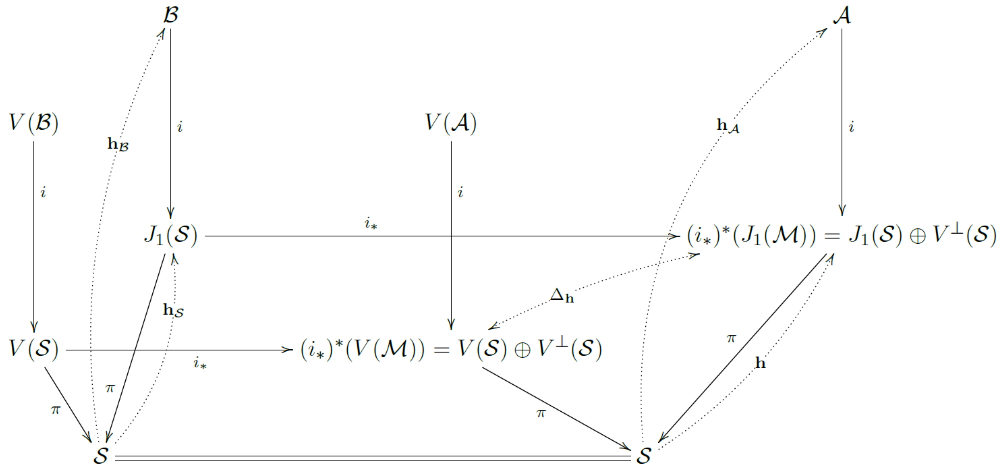

Diagram 2

and than can be considered, naively speaking, a rest frame of the corresponding constraint.

3 - Constitutive characterization of constrained impulsive systems

The impulsive problem for a constrained system has a formulation very similar to that of the impulsive problem for free system, but the mechanical foundations of the two problems are very different.

In the case of free systems, both $\mathbf{p}_L$ and $\mathbf{p}_R$ are not subject to any restriction. Then this also holds for the impulse $\mathbf{I}_{act}$, and any arbitrary rule assigning $\mathbf{I}_{act}$ as function of $\mathbf{p}_L$ is admissible. The INL is $\mathbf{p}_R = \mathbf{p}_L + \mathbf{I}_{act}(\mathbf{p}_L)$ and the principle of determinism entails that the active impulse $\mathbf{I}_{act}(\mathbf{p}_L)$ must be an a priori known data of the problem.

In the case of constrained systems, $\mathbf{p}_L$ and $\mathbf{p}_R$ are subject to the restrictions given by the constraints. Then the so called reactive impulse $\mathbf{I}_{react}$ cannot be assigned a priori as an arbitrary function of $\mathbf{p}_L$, but it is an unknown of the problem, obeying the condition that, for every admissible choice of $\mathbf{p}_L$, the right velocity $\mathbf{p}_R$ satisfies the restriction of the constraints. The INL in case of constrained system assumes then the form $\mathbf{p}_R = \mathbf{p}_L + \mathbf{I}_{act}(\mathbf{p}_L) + \mathbf{I}_{react}(\mathbf{p}_L)$.

Definition 3.1. An impulsive constitutive characterization for the constraints acting on the system is a rule determining characteristics and properties of the reactive impulse $\mathbf{I}_{react}$ in a way assuring the respect of the constraints and of the principle of determinism of Classical Mechanics.

Example 9. Let a system be subject only to a permanent kinetic constraint $\mathcal{A}$. In this case, the impulsive problem exists only if an active impulse $\mathbf{I}_{act}(\mathbf{p}_L) \in (i_*)^*(V(\mathcal{M}))$ is a priori assigned. Since both $\mathbf{p}_L, \mathbf{p}_R$ must be in $\mathcal{A}$, then the total impulse $\mathbf{I}_{react}(\mathbf{p}_L) + \mathbf{I}_{act}(\mathbf{p}_L) = \mathbf{p}_R - \mathbf{p}_L \in V(\mathcal{A})$. Therefore, in this case the constitutive characterization of $\mathcal{A}$ consists in a rule that allows to determine $\mathbf{I}_{react}(\mathbf{p}_L) \in (i_*)^*(V(\mathcal{M}))$ once $\mathbf{p}_L$ and $\mathbf{I}_{act}(\mathbf{p}_L)$ are known. Note however that, since in general $\mathbf{I}_{act}(\mathbf{p}_L) \in (i_*)^*(V(\mathcal{M})) = V(\mathcal{A}) \oplus V^{\perp}(\mathcal{A})$, then every possible constitutive characterization for $\mathcal{A}$ must obey the condition $\mathcal{P}^{\perp}_{\mathcal{A}}(\mathbf{I}_{react}(\mathbf{p}_L)) = -\mathcal{P}^{\perp}_{\mathcal{A}}(\mathbf{I}_{act}(\mathbf{p}_L))$ and then the constitutive characterization is determined by a rule assigning $\mathcal{P}^{\|}_{\mathcal{A}}(\mathbf{I}_{react}(\mathbf{p}_L))$ as function of $\mathbf{p}_L$ and $\mathbf{I}_{act}(\mathbf{p}_L)$. △

Example 10. Let a system be subject only to a unilateral positional constraint $\mathcal{S}$. Given a left velocity $\mathbf{p}_L \in \mathcal{L}_{\mathcal{S}}(J_1(\mathcal{M}))$ the impulsive problem can exists even in absence of active impulse and consists in determining the right velocity in the form $\mathbf{p}_R = \mathbf{p}_L + \mathbf{I}_{react}(\mathbf{p}_L) \in \mathcal{R}_{\mathcal{S}}(J_1(\mathcal{M}))$. The constitutive characterization of $\mathcal{S}$ consists in a rule that allows to determine univocally $\mathbf{I}_{react}(\mathbf{p}_L) \in (i_*)^*(V(\mathcal{M}))$, and then $\mathbf{p}_R$, once $\mathbf{p}_L$ is known. △

There are then two different classes of constitutive characterization: the one modelling the impulsive reaction of the constraint in absence of active impulses and the one modelling the impulsive reaction of the constraint when the system is subject to an active impulse. The first one will be the main, but not the only, focus of the following analysis.

The wide generality of the definition of constitutive characterization of constraints constitutes the fertile arena where we can model different physical behaviors of the same geometric constraint. For example, the two well known concepts of ideality and friction find their logical foundation in the context of constitutive characterization, and the same geometric constraint can be ideal or not, frictionless or not. Vice versa, the same principle inspiring a constitutive characterization can determine different effective characterizations when applied to geometrically different constraints, so that the ideality of single or multiple constraints can be performed through different rules.

Remark 7. Neglecting the forces acting on an impulsive mechanical system is an usual assumption in the context of Impulsive Mechanics. This seems only partially reasonable. In fact, if we consider the reactive forces that act on the system in the very short time interval of the impact phenomenon, their action is efficiently modelled by assuming as instantaneous the impact and introducing the reactive impulse $\mathbf{I}_{react}(\mathbf{p}_L)$. However, neglecting the influence of the active forces on the behavior of an impulsive system seems less reasonable. For instance, let us consider Ex.8: it is an experimental evidence that in the impact of the sphere with the vertical wall, the permanent kinetic constraint $\mathcal{A}$ can break, so that the permanent rolling condition verified before the impact can be not verified after the impact. Such an eventuality surely depends on the magnitude of $\mathbf{V}^{\perp}(\mathbf{p}_L)$, but it is inconceivable that it does not depend on the weight of the sphere, that tightens the contact between the sphere and the horizontal plane.

The efficacy of the geometric framework introduced above is pointed out further by showing that the active forces acting on the system can enter in the choice of the constitutive characterization of the constraint.

The second jet–extension $\pi : J_2(\mathcal{M}) \to J_1(\mathcal{M})$ of the bundle $\mathcal{M}$ (see [**MP91**]) is the *absolute acceleration space–time* of the system. It is the $(3n+1)$–dimensional affine subbundle of the tangent bundle $T(J_1(\mathcal{M}))$ whose elements have the form $\mathbf{p} = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + a^i \frac{\partial}{\partial \dot{x}^i}$, $i = 1, \ldots, n$, so that $J_2(\mathcal{M})$ can be locally described by jet–coordinates $(t, x^i, \dot{x}^i, \ddot{x}^i)$.

The affine jet–bundle $J_2(\mathcal{M}) \to J_1(\mathcal{M})$ is modelled on the $(3n+1)$–dimensional vector bundle $V(J_1(\mathcal{M})) \to J_1(\mathcal{M})$ of the vertical vectors of $T(J_1(\mathcal{M}))$, that is the vectors that are tangent to the fiber of $\pi : J_1(\mathcal{M}) \to \mathcal{M}$. Using admissible coordinates, the elements of $V(J_1(\mathcal{M}))$ have the form $\mathbf{Z} = Z^i \frac{\partial}{\partial \dot{x}^i}$, $i = 1, \ldots, n$, so that $V(J_1(\mathcal{M}))$ too can be locally described by jet–coordinates $(t, x^i, \dot{x}^i, \ddot{x}^i)$. The correspondence $\frac{\partial}{\partial x^i} \leftrightarrow \frac{\partial}{\partial \dot{x}^i}$ gives a natural isomorphism Υ of the vertical spaces $V(\mathcal{M})$ and $V(J_1(\mathcal{M}))$.

The assignment of the active forces acting on the system (see once again [**MP91**]) consists in the assignment of a section $\Theta : J_1(\mathcal{M}) \to V(J_1(\mathcal{M}))$, locally expressed by the functions $Z^i = Z^i(t, x^j, \dot{x}^j)$, $i, j = 1, \ldots, n$. Therefore, the knowledge of the active forces acting on the (impulsive) system determines the knowledge of a map $\Upsilon(\Theta(\mathbf{p})) : J_1(\mathcal{M}) \to V(\mathcal{M})$. This is enough to allow that the active forces (alternatively expressed in the form $\Upsilon(\Theta(\mathbf{p}))$ or in the local form $Z^i = Z^i(\mathbf{p})$) can be taken into account in the choice of the constitutive characterization $\mathbf{I}_{react} = \mathbf{I}_{react}(\mathbf{p}_L, \Upsilon(\Theta(\mathbf{p}_L)))$. $\diamondsuit$

In this section, we describe some of the most common constitutive characterizations for the main classes of constraints geometrically distinguished as above, mainly in absence of active impulse.

3.1 - *Ideality*

The standard requirement of ideality for a constraint acting on a system is that the reaction does not perform power or work. This cannot be required in an impulsive phenomenon, that is instantaneous and without variation of position.

3.1.1 - Ideality in presence of active impulse

The most natural requirement of ideality can repeat the standard arguments of non impulsive Classical Mechanics, for instance requiring the absence of a "tangent" component of the reactive impulse.

Example 11. Let $\mathcal{S}$ a bilateral positional constraint and let $\mathbf{I}_{act}(\mathbf{p}_L) \in (i_*)^*(V(\mathcal{M})) = V(\mathcal{S}) \oplus V^{\perp}(\mathcal{S})$ be such that $\mathcal{P}^{\perp}(\mathbf{I}_{act}(\mathbf{p}_L)) \neq 0$. This is a case of system subject to a permanent positional constraint whose presence cannot be absorbed in the construction itself of the space–time bundle $\mathcal{M}$ for the nature itself of the impulsive problem (see 2.1.1).

With a slight abuse of notation and recalling Ex.9, a possible ideal criterion is given in this case by the conditions $\mathcal{P}^{\perp}(\mathbf{I}_{react}(\mathbf{p}_L)) = -\mathcal{P}^{\perp}(\mathbf{I}_{act}(\mathbf{p}_L))$ and $\mathcal{P}^{\|}(\mathbf{I}_{react}(\mathbf{p}_L)) = 0$.

The analogous choice $\mathcal{P}^{\|}_{\mathcal{A}}(\mathbf{I}_{react}(\mathbf{p}_L)) = 0$ can express (see once again Ex.9) the ideality of a permanent kinetic constraint $\mathcal{A}$ in presence of a generic active impulse $\mathbf{I}_{act}(\mathbf{p}_L) \in (i_*)^*(V(\mathcal{M})) = V(\mathcal{A}) \oplus V^{\perp}(\mathcal{A})$. $\triangle$

3.1.2 - Ideality in absence of active impulse

The most natural requirement of ideality consists in the preservation of the kinetic energy of the system before and after the impulsive action of the constraint. However a naive formulation of this requirement determines obvious inconsistencies of the approach, even for very simple mechanical system.

Example 12. We consider a particular impulsive motion of a rod of length $2L$ and mass M moving in a half–plane. The space–time bundle can be described by coordinates (t, x, y, ϑ) where (x, y) are the cartesian coordinates of the center of the rod and ϑ is its orientation. The vertical metric is expressed by the mass matrix $g_{ij} = diag(M, M, A)$ with $A = \frac{1}{3}ML^2$. The unilateral constraint $\mathcal{S}$ can be locally described by the condition $y - L\sin\vartheta = 0$. Using admissible coordinates we suppose that the impact happens in the point $(t_0, x_0, L, \pi/2)$ with absolute velocities

$$\mathbf{p}_L = \frac{\partial}{\partial t} - \dot{y}_0 \frac{\partial}{\partial y}; \qquad \mathbf{p}_R = \frac{\partial}{\partial t} + \dot{y}_0 \frac{\partial}{\partial y}.$$

Then, roughly speaking, the rod, moving downward in vertical position with vertical velocity $-\dot{y}_0$ respect to constraint, impacts with the constraint and rebounds in vertical position with vertical velocity $\dot{y}_0$. Intuitively, the impact seems to have an ideal behavior, since the kinetic energy seems preserved in the impact. However, as we already saw, this is a meaningless statement until we do not specify the frame of reference where the kinetic energy is preserved. In fact, if we introduce the three frames of reference:

$$\mathbf{h}_0 = \frac{\partial}{\partial t}; \qquad \mathbf{h}_1 = \frac{\partial}{\partial t} + \dot{y}_0 \frac{\partial}{\partial y}; \qquad \mathbf{h}_2 = \frac{\partial}{\partial t} - \dot{y}_0 \frac{\partial}{\partial y},$$

the kinetic energy of the system in the impact is preserved for $\mathbf{h}_0$, decreases to zero for $\mathbf{h}_1$ and even increases from zero to a positive value for $\mathbf{h}_2$. This is (obviously) due to the facts that $\mathbf{h}_0$ is a rest frame for the constraint, $\mathbf{h}_1$ is a "comoving" frame of the rod after the impact and $\mathbf{h}_2$ is a "comoving" frame of the rod before the impact. △

The example above shows that, of course, preservation of kinetic energy cannot be required for all the frames of reference of the system. On the other side, requiring the preservation in a single frame conflicts with the basic requirement that a physical property of a system must be independent of the frame of reference.

It is a known result that the requirement of preservation of the kinetic energy for all the frames of reference in the class of the rest frames of the constraints is sufficient (in absence of active impulses) to determine a satisfactory constitutive characterization for some significant classes of constraints, such as unilateral constraints of codimension 1 (see [**Pas05a**]), also in presence of permanent and/or instantaneous kinetic constraints (see [**Pas06**]). In these cases, roughly speaking, the impulsive reaction is determined by the orthogonal component $\mathbf{V}^{\perp}(\mathbf{p}_L)$ of the left–velocity in the form $\mathbf{I}_{react}(\mathbf{p}_L) = -2\mathbf{V}^{\perp}(\mathbf{p}_L)$, and the impulsive behavior of the system is essentially a reflection with respect to the "orthogonal direction" determined by the constraint.

R e m a r k 8. The impact of Ex.12 is really ideal since the kinetic energy is preserved for all the rest frames of the constraint $y - L\sin\vartheta = 0$ (and in particular for $\mathbf{h}_0$). $\diamondsuit$

It is also known that the only preservation of kinetic energy is not sufficient to determine univocally the constitutive characterization for some classes of constraints (such as positional constraints of codimension greater than 1 and multiple unilateral constraints). Nevertheless, the "reflection" characterization $\mathbf{I}_{react}(\mathbf{p}_L) = -2\mathbf{V}^{\perp}(\mathbf{p}_L)$ could be (such as in the case of positional constraints of codimension greater than 1. See once again [**Pas05a**]) or at least could suggest (such as in the case of multiple unilateral constraints. See [**Pas18a**]) physically meaningful ideal constitutive characterizations.

R e m a r k 9. Of course the requirement of preservation of the kinetic energy is in general not admissible in presence of active impulses, as clearly shown by Ex.11. $\diamondsuit$

Needless to say, the importance of the ideal (and in particular the "reflection") characterization relies not only in its structural simplicity (it involves only the geometric structures determined by the constraints) and its effective applicability to several meaningful systems, but also because it constitutes the starting point to analyze non ideal behaviors of constraints.

3.2 - *Frictionless non–ideality*

The ideal requirement of preservation of kinetic energy in absence of active impulse suggests the possible non–ideal characterization where the kinetic energy is partially or totally lost in a non–elastic impact. Once again, the naive idea of a fixed percentage of loss of kinetic energy, expressed for example by an energetic restitution coefficient $\varepsilon_{\mathcal{K}} \in [0,1)$, gives rise to inconsistencies.

E x a m p l e 13. The same rod of Ex.12 vertically falls but does not rebound on the constraint, so that its time evolution after the impact is given by the motion $\gamma(t) = (t, x_0, L, \pi/2)$ and $\mathbf{p}_R = \frac{\partial}{\partial t}$. By choosing $\mathbf{h}_{\mathcal{S}} = \frac{\partial}{\partial t} + H_x \frac{\partial}{\partial x} \in \mathcal{H}_{\mathcal{S}}$ for every H_x, with obvious notation we have that

$$\varepsilon_{\mathcal{K}} = \frac{(\mathcal{K}_{\mathbf{h}_0})_R}{(\mathcal{K}_{\mathbf{h}_0})_L} = \frac{(H_x)^2}{(H_x)^2 + (\dot{y}_0)^2},$$

that explicitly depends on H_x. $\triangle$

The example above shows that the percentage of loss of kinetic energy in a non–elastic impact is not frame invariant even if we restrict the assignment of $\varepsilon_{\mathcal{K}}$ to the class of rest frames of the constraint. A frame invariant formulation for the restitution coefficient can be obtained by considering the percentage of "reflection" of the orthogonal velocity in the impact: a (Newtonian) coefficient $\varepsilon^{\perp} \in [0,1)$ such that

$$\mathbf{I}_{react}(\mathbf{p}_L) = -(1+\varepsilon^{\perp})\,\mathbf{V}^{\perp}(\mathbf{p}_L)$$

has a clear invariant (with respect to the rest frames) meaning.

The particular case $\varepsilon^{\perp} = 0$ naturally gives the non–ideal *totally inelastic* characterization, defined as

$$\begin{array}{rccl} TotIn: & \mathcal{L}_{\mathcal{S}}(J_1(\mathcal{M})) & \to & (i_*)^*(V(\mathcal{M})) \qquad \text{s.t.} \\ & \mathbf{p}_L & \rightsquigarrow & \mathbf{I}_{TotIn}(\mathbf{p}_L) = -\mathbf{V}^{\perp}(\mathbf{p}_L)\,. \end{array}$$

The evolution equation assumes the form

$$\begin{array}{ccl} \mathcal{L}_{\mathcal{S}}(J_1(\mathcal{M})) & \to & \mathcal{R}_{\mathcal{S}}(J_1(\mathcal{M})) \qquad \text{s.t.} \\ \mathbf{p}_L & \rightsquigarrow & \mathbf{p}_R = \mathbf{p}_L - \mathbf{V}^{\perp}(\mathbf{p}_L) = \mathcal{P}_{\mathcal{S}}(\mathbf{p}_L)\,. \end{array}$$

Note that, in this case, there exists a subclass of the rest frames $\mathcal{H}_{\mathcal{S}}$ of $\mathcal{S}$ formed by all those frames such that $\mathbf{h}(\pi(\mathbf{p}_L)) = \mathcal{P}_{\mathcal{S}}(\mathbf{p}_L)$ for which the system stops after the impact and then have null kinetic energy. Of course this property does not hold for all the frames of $\mathcal{H}_{\mathcal{S}}$.

3.3 - *Friction*

The naive idea of impulsive constraint with friction can be expressed by the condition $\mathcal{P}^{\parallel}(\mathbf{I}_{react}(\mathbf{p}_L)) \neq 0$. Once again, due to the wide variety of possible constraints acting on impulsive systems, there are several different way to apply the naive idea sketched above to the various constraints.

3.3.1 - Friction in presence of active impulse

Given a system subject to a bilateral positional constraint $\mathcal{S}$ (resp. a system subject to a kinetic permanent constraint $\mathcal{A}$) and subject to an active impulse such as $\mathbf{I}_{act} \in (i_*)^*(V(\mathcal{M}))$ and $\mathbf{I}_{act} \notin V(\mathcal{S})$ (resp. $\mathbf{I}_{act} \in (i_*)^*(V(\mathcal{M}))$ and $\mathbf{I}_{act} \notin V(\mathcal{A})$), the standard arguments of Classical (non Impulsive) Mechanics can be applied replacing active and reactive forces with active and reactive impulses: the projection operators $\mathcal{P}^{\parallel}, \mathcal{P}^{\perp}$ determine the tangent and parallel components of the active impulse and a constitutive characterization (for example of Coulomb type) of the reactive impulse can be assigned as functions of these components.

Example 14. A massive particle moving on a horizontal floor is subject to an active impulse $\mathbf{I}_{act}$ that is not parallel to the floor. The 4–dimensional space–time $\mathcal{M}$ can be described by the coordinates (t, x, y, z), where (x, y, z) are the coordinates of the particle, together with the permanent positional constraint $\mathcal{S} = \{z = 0\}$. The active impulse $\mathbf{I}_{act} \in (i_*)^*(V(\mathcal{M}))$ but $\mathbf{I}_{act} \notin V(\mathcal{S})$.

Once again, since both $\mathbf{p}_L, \mathbf{p}_R \in J_1(\mathcal{S})$, the total impulse $\mathbf{I}_{act}(\mathbf{p}_L) + \mathbf{I}_{react}(\mathbf{p}_L) = \mathbf{p}_R - \mathbf{p}_L \in V(\mathcal{S})$. Then the condition $\mathcal{P}^{\perp}(\mathbf{I}_{react})) = -\mathcal{P}^{\perp}(\mathbf{I}_{act})$ is mandatory in order to respect the constraint. We can then introduce a constitutive characterization with friction for $\mathcal{S}$ by assigning $\mathcal{P}^{\parallel}(\mathbf{I}_{react}(\mathbf{p}_L))$ as a non–null function of $(\mathbf{p}_L, \mathcal{P}^{\parallel}(\mathbf{I}_{act}(\mathbf{p}_L)), \mathcal{P}^{\perp}(\mathbf{I}_{act}(\mathbf{p}_L)))$. △

Example 15. A massive particle moving in a 3–dimensional euclidean space is subject to a permanent kinetic constraint $\mathcal{A}$ and to an active impulse $\mathbf{I}_{act}(\mathbf{p}_L) \in (i_*)^*(V(\mathcal{M})), \mathbf{I}_{act}(\mathbf{p}_L) \notin V(\mathcal{A})$. Since the total impulse $\mathbf{I}_{act}(\mathbf{p}_L) + \mathbf{I}_{react}(\mathbf{p}_L) = \mathbf{p}_R - \mathbf{p}_L \in V(\mathcal{A})$, then $\mathcal{P}^{\perp}_{\mathcal{A}}(\mathbf{I}_{react}(\mathbf{p}_L)) = -\mathcal{P}^{\perp}_{\mathcal{A}}(\mathbf{I}_{act}(\mathbf{p}_L))$ and a constitutive characterization of $\mathcal{A}$ can be assigned determining $\mathcal{P}^{\parallel}_{\mathcal{A}}(\mathbf{I}_{react}(\mathbf{p}_L))$ as a non–null function of $(\mathbf{p}_L, \mathcal{P}^{\parallel}_{\mathcal{A}}(\mathbf{I}_{act}(\mathbf{p}_L)), \mathcal{P}^{\perp}_{\mathcal{A}}(\mathbf{I}_{act}(\mathbf{p}_L)))$. △

3.3.2 - Friction in absence of active impulse

For impulsive systems that are not subject to active impulses, the analogies between classical and impulsive description of friction is in general not possible. This is mainly due to geometrical reasons.

Let $\mathcal{S}$ be a unilateral positional constraint: the condition $\mathcal{P}^{\parallel}(\mathbf{I}_{react}(\mathbf{p}_L)) \neq 0$ can be taken as starting point to define a constitutive characterization with friction for $\mathcal{S}$, but the effective applicability of this idea must, in general, take into account the lack of prior specific directions in $V(\mathcal{S})$. The system of Ex.3 gives the simplest example of this situation.

The usual (and sometimes unaware) way to select a prior direction in $V(\mathcal{S})$ consists in the assignment of a frame of reference $\mathbf{h}_{\mathcal{S}} \in \mathcal{H}_{\mathcal{S}}$ thought of as "the" rest frame of the constraint. In fact, in this case, we can consider the tangent component $\mathbf{V}^{\parallel}_{\mathbf{h}_{\mathcal{S}}}(\mathbf{p}_L) = \mathcal{P}_{\mathcal{S}}(\mathbf{p}_L) - \mathbf{h}_{\mathcal{S}} = \mathcal{P}^{\parallel}(\mathbf{p}_L - \mathbf{h}_{\mathcal{S}})$ of the velocity, and we can choose $\mathcal{P}^{\parallel}(\mathbf{I}_{react}(\mathbf{p}_L)) \in Lin\{\mathbf{V}^{\parallel}_{\mathbf{h}_{\mathcal{S}}}(\mathbf{p}_L)\}$. The constitutive characterization can then be assigned in the form

$$(8)\qquad \begin{cases} \mathcal{P}^{\perp}(\mathbf{I}_{react}(\mathbf{p}_L)) \;=\; \alpha(\mathbf{p}_L)\,\mathbf{V}^{\perp}_{\mathbf{h}_{\mathcal{S}}}(\mathbf{p}_L) \\[2ex] \mathcal{P}^{\|}(\mathbf{I}_{react}(\mathbf{p}_L)) \;=\; \beta(\mathbf{p}_L)\,\mathbf{V}^{\|}_{\mathbf{h}_{\mathcal{S}}}(\mathbf{p}_L) \end{cases}$$

for suitable choices of the functions $\alpha(\mathbf{p}_L), \beta(\mathbf{p}_L)$.

Remark 10. The presence of a non–null frictional component $\mathcal{P}^{\|}(\mathbf{I}_{react}(\mathbf{p}_L)) \neq 0$ does not imply the non–ideality of the characterization of the constraint. The ideal characterization of the set of constraints comprised of a unilateral positional constraint $\mathcal{S}$ and an instantaneous kinetic constraint $\mathcal{B}$, due to the splitting $V(\mathcal{M}) = V(\mathcal{B}) \oplus V^{\perp}(\mathcal{B}) \oplus V^{\perp}(\mathcal{S})$ (see [**Pas06**]) gives an example of ideal constitutive characterization with friction. $\diamondsuit$

3.4 - *Breakability*

The naive idea of systems subject to a breakable impulsive constraint is that of systems whose behavior respects the constraint condition before the impulsive phenomenon and does not respect the constraint condition after the impulsive phenomenon. There are many physical examples of this possibility: for instance, a bullet that can bounce on or perforate a wall, or a billiard ball that rolls on the table before the impact with the cushion and that slides after the impact.

In order to give a geometrical description of breakability of a positional constraint $\mathcal{S}$, we introduce a generalization of the spaces $\mathcal{L}_{\mathcal{S}}(J_1(\mathcal{M})), \mathcal{R}_{\mathcal{S}}(J_1(\mathcal{M}))$ of Section 2.1: we say that $\mathcal{S}$ is breakable if $\mathcal{L}_{\mathcal{S}}(J_1(\mathcal{M})) \cap \mathcal{R}_{\mathcal{S}}(J_1(\mathcal{M})) \neq \emptyset$. This simply means that there are some admissible left velocities that can also be admissible right velocities.

Example 16. Let $\mathcal{S}$ be a positional constraint of codimension 1, let $\mathcal{L}_{\mathcal{S}}(J_1(\mathcal{M})) = \{\mathbf{p}_L \,|\, \Phi(\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L), \mathbf{U}^{\perp}) < 0\}$ and let the constitutive characterization of $\mathcal{S}$ be such that $\mathbf{I}_{react}(\mathbf{p}_L) = -\,\lambda_{\Xi}(\mathbf{p}_L)\,\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)$ with

$$\lambda_{\Xi}(\mathbf{p}_L) \;=\; 2\,\frac{\Xi^2}{\Xi^2 + \|\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)\|^2}\,, \qquad \Xi > 0\,.$$

In this case, $\mathcal{S}$ is "almost" elastic for $\|\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)\| \ll 1$, it is anelastic for $\|\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)\| < \Xi$, it is totally anelastic for $\|\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)\| = \Xi$, it is broken for $\|\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)\| > \Xi$. In this case the value Ξ represents the "breakability threshold" of $\mathcal{S}$.

A similar characterization gives, for example, a model for the impulsive behavior of a bulletproof glass. $\triangle$

E x a m p l e 17. Let $\mathcal{S}$ and $\mathcal{L}_{\mathcal{S}}(J_1(\mathcal{M}))$ be as above and let the constitutive characterization of $\mathcal{S}$ be such that $\mathbf{I}_{react}(\mathbf{p}_L) = -\lambda_\Xi(\mathbf{p}_L)\,\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)$ with

$$\lambda_\Xi(\mathbf{p}_L) = 2\,\frac{\|\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)\|^2}{\Xi^2 + \|\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)\|^2}\,, \qquad \Xi > 0\,.$$

In this case, $\mathcal{S}$ is "almost" elastic for $\|\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)\| \gg 1$, it is totally anelastic for $\|\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)\| = \Xi$, it is broken for $\|\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)\| < \Xi$. Once again the value Ξ represents the "breakability threshold" of $\mathcal{S}$.

A similar characterization gives, for example, a model for the impulsive behavior of the surface of a non–newtonian fluid. △

An impulsive action (for instance, an active impulse or the impact with a unilateral positional constraint $\mathcal{S}$) can break the action of a kinetic "permanent" constraint $\mathcal{A}$. In this case the breakability of $\mathcal{A}$ can be modelled by assuming $\mathcal{A}$ or $\mathcal{L}_{\mathcal{S}}(\mathcal{A})$ as space of admissible left velocities and $J_1(\mathcal{M})$ or $\mathcal{R}_{\mathcal{S}}(J_1(\mathcal{M}))$ as space of admissible right velocities.

E x a m p l e 18. The sphere of Ex.8 rolling without sliding on the horizontal plane and impacting with the vertical wall is subject to the following constitutive characterization:

$$\mathbf{I}_{react}(\mathbf{p}_L) = \begin{cases} -(1+\varepsilon_1)\,\mathbf{V}^{\perp}_{J_1(\mathcal{S})\cap\mathcal{A}}(\mathbf{p}_L) & \text{if} \quad \|\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p})\| \le \Xi \\ \\ -(1+\varepsilon_2)\,\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L) & \text{if} \quad \|\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p})\| > \Xi \end{cases}$$

where $\varepsilon_1, \varepsilon_2 \in [0,1]$ and $\Xi \ge 0$. Of course, the constants $\varepsilon_1, \varepsilon_2$ represent the restitution coefficient of the contact sphere/wall and Ξ represents the breakability threshold of the pure rolling kinetic constraint. Note moreover that, since the computation of $\mathbf{V}^{\perp}_{\mathcal{S}}(\mathbf{p}_L)$ involves the mass M of the disk, for suitable choices of Ξ a similar constitutive characterization can embody the weight force acting on the sphere and tightening the contact between sphere and horizontal plane. △

4 - Conclusions

Pursuing the aim of providing a clarifying organization of the operational and applicable bases of Impulsive Mechanics of constrained systems, we showed that the geometric setup given by the space–time configuration bundle $\pi_t : \mathcal{M} \to \mathbb{E}$ and its jet extensions and subbundles, as well as neatly framing Classical "smooth" Mechanics with its axioms and invariance requirements, similarly provides the geometric context where Classical Impulsive Mechanics too, with its own causal structure and invariance requirements, can find a natural setting. In particular it allows:

- an invariant and causally correct formulation of the INL both for free and constrained systems such that, possibly through the concept of constitutive characterization, respects the determinism of Classical Mechanics;
- a clear distinction of the different classes of constraints, both impulsive and not, depending on their geometric properties;
- a clear distinction of the different behavior of impulsive constraints, depending on their constitutive characterization.

Along the paper, we had the opportunity of emphasizing that:

- the geometric environments given by the configuration space $\mathcal{Q}$ or the product bundles $\mathbb{R} \times \mathcal{Q}$ and $\mathbb{E} \times \mathcal{Q}$, because of their intrinsic selection of a frame of reference, are not appropriate to obtain the frame invariance, and that this is particularly damaging in an impulsive context;
- the concept of friction can be introduced in the constitutive characterization of a unilateral positional constraint only if a rest frame of the constraint is chosen.

Moreover, we also introduced and discuss some unusual but meaningful concepts and behaviors of impulsive systems:

- the concept of unilateral kinetic (permanent or instantaneous) constraint;
- the concept of breakable impulsive constraint;
- the theoretical possibility that the active forces acting on the system are involved in the definition of the impulsive constitutive characterization of an impulsive constraint.

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