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Space-like Willmore immersions

Abstract. In this survey paper we implement the method of moving frames to the Lorentzian setting. As an application, we are able to give a geometrical proof of the fact that a space-like Willmore immersion of a compact surface in the oriented, time-oriented conformal compactification of the Minkowski 3-space must be totally umbilical.

Keywords. Conformal Lorentzian geometry, Einstein Universe, space-like Willmore immersions.

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1 - Introduction

Our interest in Lorentzian conformal geometry is motivated by the conformally cyclic cosmological models proposed in the last decade by R. Penrose [**23**,**24**,**27**]. In these models the universe undergoes to a cyclical evolution which is smooth for what concerns the conformal structure of the space-time but develops singularities from the viewpoint of the Lorentzian structure. The space-time is subdivided into "eons", i.e. adjacent portions of the space-time provided with a smooth Lorentzian pseudo-metric belonging to the given conformal class. They are bounded by conformally flat space-like hypersurfaces at which the pseudo-metric tensor becomes singular, the conformal boundaries of the "eon". In the proximity of the conformal boundaries, the conformal structure remains smooth and is modeled on that of the static Einstein universe. It is then reasonable to impose some kind of conformally invariant variational principle to constrain the geometry of the conformal boundaries. A natural choice

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is given by the Blaschke energy, originally investigated by Blaschke and Thomsen in the 3-dimensional Riemannian framework. In the current literature the Blaschke energy is known as the Willmore functional and its critical points are referred to as Willmore immersions (M-minimal in the classical terminology). They have been the focus of intense research in the last thirty years mainly in connection with the Willmore conjecture, recently solved by Fernando Codá Marques and André Neves, [**20**].

The study of Willmore surfaces in the Lorentzian context is not as popular as that in the Riemannian framework. However it is not completely unexplored (see for instance [**1**, **2**, **8**, **21**, **22**, **28**]). From the perspective of this paper, the more useful results are those obtained by Alías and Palmer in [**1**], where, among other properties, the authors generalize in the Lorentzian setting two facts that have been originally proved by G. Thomsen, [**26**], in the context of classical Möbius geometry : they define the conformal Gauss map of a space-like immersed surface, they prove that the conformal Gauss map is conformal and that the immersion fulfills the variational equation of the Willmore functional if and only if its conformal Gauss map is harmonic. We will use these facts during our investigations.

Inspired by a paper of R. Bryant, [**6**], about Willmore surfaces in the conformal 3-sphere, in the present paper we aim at implementing the method of moving frames in the Lorentzian setting and at showing how this apparatus can be used to investigate the geometry of space-like Willmore (or M-maximal) immersed surfaces in the 3-dimensional oriented, time-oriented conformal Einstein universe. In particular, we will use this method to recover, in a more geometrical way, some results already known in literature.

Before stating the main results, we recall that the physical Einstein universe is the cartesian product $\mathcal{E}^* = \mathbb{R} \times \mathbb{S}^2$ equipped with the Lorentzian metric $-\mathrm{d}t^2 + g_{\mathbb{S}^2}$. It was proposed by Einstein himself as the first example of a static space-time with a positive cosmological constant, [**11**]. The group Γ generated by the translation $(t, \mathrm{x}) \to (t + 2\pi, \mathrm{x})$ acts properly discontinuously on $\mathcal{E}^*$. The quotient $\mathcal{E} = \mathcal{E}^*/\Gamma$ equipped with the induced Lorentzian structure is referred to as the oriented, time-oriented compact model of the Einstein universe [**3**, **14**, **15**]. As an application of the aforementioned method of moving frames, we will obtain the following theorems (the first one originally due to Alías and Palmer [**1**]):

Theorem A. *Let $\mathcal{S}$ be a connected, compact 2-dimensional manifold and $f : \mathcal{S} \to \mathcal{E}$ be a space-like Willmore immersion. Then, $\mathcal{S}$ is diffeomorphic to the 2-sphere and $f(\mathcal{S})$ is a totally umbilical round 2-sphere embedded in $\mathcal{E}$.*

Theorem B. *Let $\mathcal{S}$ be a connected, compact 2-dimensional manifold and $f : \mathcal{S} \to \mathcal{E}^*$ be a space-like Willmore immersion. Then, $\mathcal{S}$ is diffeomorphic to* *the* 2*-sphere and* $f(\mathcal{S})$ *is a totally umbilical round* 2*-sphere embedded in* $\mathcal{E}^*$*.*

The paper is organized into three Sections and one Appendix. In the first Section we recall the basics about the conformal geometry of the oriented, time-oriented conformal compactification $\mathcal{E}$ of the Minkowski 3-space. In the second Section we present the background material on the conformal geometry of a space-like immersed surface $f : \mathcal{S} \to \mathcal{E}$ by using from the outset a conformally invariant approach based on the method of moving frame, [**17**]. We recall the construction of the conformal Gauss map and we define the fundamental conformal invariants of a space-like surface: the Blaschke's linear and area elements and the Bryant's quartic differential. Subsequently we introduce the notion of second-order conformal frame and second-order frames adapted to a complex chart. Adapted frames are used to build the Poynting field and the dual map of a space-like Willmore immersion. The third Section is devoted to the proofs of Theorem A and Theorem B. In the Appendix we verify a technical elementary property, used in the third Section, about complex-valued functions of analytic type. This result appears, without proof, in the existing literature (see for instance [**6**, **18**]).

2 - Conformal Lorentzian geometry

2.1 - *The oriented, time-oriented compact model of the Einstein universe*

Let $\mathbb{R}^{2,3}$ denote the vector space $\mathbb{R}^5$ equipped with the non-degenerate scalar product of signature $(2,3)$ defined by

$$\langle \mathrm{V}, \mathrm{W}\rangle = -(\mathrm{v}^0\mathrm{w}^4 + \mathrm{v}^4\mathrm{w}^0) - \mathrm{v}^1\mathrm{w}^1 + \mathrm{v}^2\mathrm{w}^2 + \mathrm{v}^3\mathrm{w}^3 = {}^{\mathrm{t}}\,\mathrm{V}\cdot \mathtt{m}\cdot \mathrm{W}, \tag{2.1}$$

$\mathtt{m} = (\mathtt{m}_{ij})$, $\mathtt{m}_{ij} = \mathtt{m}_{ji}$, $i,j = 0,...,4$ and with the orientation induced by the volume form $\mathcal{V} = \mathrm{dv}^0 \wedge ... \wedge \mathrm{dv}^4$. Unlike the Lorentzian case, it is not possible to attribute a causal character to null or time-like oriented lines. Instead, the notion of time-orientation can be defined for causal planes.

Definition 2.1. A *causal plane* is a 2-dimensional oriented[1] vector subspace $[\mathbb{V}] \subset \mathbb{R}^{2,3}$ such that $\langle \mathrm{Y}, \mathrm{Y}\rangle \leq 0$, for all $\mathrm{Y} \in \mathbb{V}$. If V and W are linearly independent vectors, the oriented plane spanned by (V, W) is denoted by $|[\mathrm{V}\wedge\mathrm{W}]|$. To define the character of a causal plane we pick the negative-definite oriented 2-plane $[\mathbb{V}_*] = |[(\mathrm{E}_0+\mathrm{E}_4)\wedge\mathrm{E}_1]|$, where $\mathbf{E} = (\mathrm{E}_0, ..., \mathrm{E}_4)$ is the standard basis of $\mathbb{R}^{2,3}$. Let $\pi_* : \mathbb{R}^{2,3} \to \mathbb{V}_*$ be the orthogonal projection onto $\mathbb{V}_*$. For each causal plane $[\mathbb{V}]$ the linear map $\pi_*|_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}_*$ is an isomorphism. We say

[1] We use the notation $[\mathbb{V}]$ for a vector subspace $\mathbb{V} \subset \mathbb{R}^{2,3}$ endowed with an orientation.

that $[\mathbb{V}]$ is *future-oriented* if $\pi_*|_{\mathbb{V}}$ is orientation-preserving. The semi-analytic set of all future-directed (resp. past-directed) causal planes is denoted by $\mathcal{L}^{\uparrow}$ ($\mathcal{L}^{\downarrow}$ resp.) .

Definition 2.2. The (restricted) *automorphism group* of $\mathbb{R}^{2,3}$, denoted by $\mathrm{M}^{\uparrow}_{+}$, is the connected component of the identity of the pseudo-orthogonal group of $\mathbb{R}^{2,3}$, that is, the Lie group of all linear isometries of $\mathbb{R}^{2,3}$ preserving the orientation and the time-orientation of causal planes. Its Lie algebra $\mathfrak{m}$ is the vector space of all skew-adjoint endomorphisms of $\mathbb{R}^{2,3}$ with the commutator as a Lie bracket.

Notation 1. For every $\mathrm{V} \in \mathbb{R}^{2,3}$, we write $\mathrm{V} = \mathrm{V}' + \mathrm{V}''$, where $\mathrm{V}' \in \mathbb{V}_*$ and $\mathrm{V}'' \in \mathbb{V}_*^{\perp}$. We denote by $\mathbb{S}^1 \subset \mathbb{V}_*$ the circle of the unit time-like vectors belonging to $\mathbb{V}_*$ and by $\mathbb{S}^2 \subset \mathbb{V}_*^{\perp}$ the unit-sphere of $\mathbb{V}_*^{\perp}$. Let $\mathbb{J} : \mathbb{V}_* \to \mathbb{V}_*$ be the counterclockwise rotation by a straight angle in the oriented plane $[\mathbb{V}_*]$ with respect to the negative-definite scalar product $\langle -, - \rangle|_{\mathbb{V}_* \times \mathbb{V}_*}$. Given any vector V, we denote by $|[\mathrm{V}]|$ the *ray* (i.e. the oriented line) spanned by V. The manifold G_1^+ of the rays of $\mathbb{R}^{2,3}$ is diffeomorphic to the 4-dimensional sphere.

Definition 2.3. The oriented, time-oriented, *compact model* of the 3-dimensional Einstein universe is the codimension-two submanifold $\mathbb{S}^1 \times \mathbb{S}^2$ of $\mathbb{R}^{2,3}$, that is the set of all $\mathrm{V} \in \mathbb{R}^{2,3}$ such that $\langle \mathrm{V}', \mathrm{V}' \rangle = -1$ and $\langle \mathrm{V}'', \mathrm{V}'' \rangle = 1$. On $\mathbb{S}^1 \times \mathbb{S}^2$ we fix the Lorentzian metric $\ell_{\mathcal{E}}$ induced by the scalar product $\langle -, - \rangle$ and the orientation defined by the contraction of the volume form $\mathcal{V}$ with the unit normal vector fields $\mathbf{n}'|_{\mathrm{V}} = \mathrm{V}'$ and $\mathbf{n}''|_{\mathrm{V}} = \mathrm{V}''$. We define a time-orientation by requiring that the unit time-like tangent vector field $\mathbf{T}|_{\mathrm{V}} = \mathbb{J}(\mathrm{V}')$ is future-directed.

Definition 2.4. The map $j_{\mathcal{E}} : \mathrm{V} \in \mathbb{S}^1 \times \mathbb{S}^2 \to |[\mathrm{V}]| \in \mathrm{G}_1^+$ is a smooth embedding of $\mathbb{S}^1 \times \mathbb{S}^2$ into the manifold of the rays. Its image is the submanifold $\mathcal{E} \subset \mathrm{G}_1^+$ of all *null* (*isotopic, ligth-like*) rays of $\mathbb{R}^{2,3}$. This allow us to identify $\mathbb{S}^1 \times \mathbb{S}^2$ with $\mathcal{E}$ and we transfer to $\mathcal{E}$ the oriented, time-oriented conformal Lorentzian structure of $\mathbb{S}^1 \times \mathbb{S}^2$. We will make no distinction between the two models and the context will make clear which one of them is being used.

Using the above identification, the automorphism group $\mathrm{M}^{\uparrow}_{+}$ acts effectively and transitively on the left of $\mathcal{E}$ by $\mathfrak{B} \cdot |[\mathrm{V}]| = |[\mathfrak{B}(\mathrm{V})]|$, for each $\mathfrak{B} \in \mathrm{M}^{\uparrow}_{+}$ and $|[\mathrm{V}]| \in \mathcal{E}$. The action preserves the oriented, time-oriented conformal structure of $\mathcal{E}$. It is a classical result [**9**, **13**, **14**] that each restricted conformal transformation of $\mathcal{E}$ is induced by a unique element of $\mathrm{M}^{\uparrow}_{+}$.

Definition 2.5. A *Möbius basis* $\mathbf{B} = (\mathrm{B}_0, ..., \mathrm{B}_4)$ of $\mathbb{R}^{2,3}$ is a positive-oriented basis such that

$$\langle \mathrm{B_i}, \mathrm{B_j} \rangle = \mathtt{m}_{\mathrm{ij}}, \quad 0 \leq \mathrm{i}, \mathrm{j} \leq 4, \quad |[\mathrm{B}_0 \wedge \mathrm{B}_1]| \ \in \mathcal{L}^{\uparrow}.$$

Remark 2.6. Given $\mathfrak{B} \in \mathrm{M}^{\uparrow}_{+}$ let $\mathbf{B}(\mathfrak{B})$ be the matrix representing $\mathfrak{B}$ with respect to the standard basis. The map $\mathfrak{B} \in \mathrm{M}^{\uparrow}_{+} \mapsto \mathbf{B}(\mathfrak{B}) \in \mathrm{SL}(5,\mathbb{R})$ is a faithful matrix representation through which we can identify $\mathrm{M}^{\uparrow}_{+}$ with the closed subgroup of all matrices $\mathbf{B} \in \mathrm{SL}(5,\mathbb{R})$ whose column vectors $\mathrm{B}_0, ..., \mathrm{B}_4$ constitute a Möbius basis.

2.2 - *Conformal embedding of the Minkowski 3-space*

Let $\mathbb{R}^{1,2}$ be the affine Minkowski 3-space, i.e. $\mathbb{R}^3$ with coordinates $(\mathtt{x}^1, \mathtt{x}^2, \mathtt{x}^3)$, oriented by the volume form $d\mathtt{x}^1 \wedge d\mathtt{x}^2 \wedge d\mathtt{x}^3$, equipped with the Lorentzian inner product

$$(\mathtt{X}, \mathtt{Y}) = -\mathtt{x}^1\mathtt{y}^1 + \mathtt{x}^2\mathtt{y}^2 + \mathtt{x}^3\mathtt{y}^3 \tag{2.2}$$

and with the time-orientation defined by the positive light-cone

$$\{\mathtt{X} \in \mathbb{R}^{1,2} : (\mathtt{X}, \mathtt{X}) = 0, \mathtt{x}^1 > 0\}.$$

Let $\mathrm{P}^{\uparrow}_{+}$ be the (restricted) Poincaré group of $\mathbb{R}^{1,2}$, i.e. the semi-direct product $\mathbb{R}^{1,2} \times_{\iota} \mathrm{O}^{\uparrow}_{+}(1,2)$ of $\mathbb{R}^{1,2}$ with the connected component of the identity of $\mathrm{O}(1,2)$. We may think of $\mathrm{P}^{\uparrow}_{+}$ as the closed subgroup of $\mathrm{SL}(4,\mathbb{R})$ consisting of all 4×4 matrices of the form

$$\mathtt{B} := \mathtt{B}(\mathtt{b}, \mathtt{X}) = \begin{bmatrix} 1 & 0 \\ \mathtt{b} & \mathtt{X} \end{bmatrix},$$

where $\mathtt{X} \in \mathrm{O}^{\uparrow}_{+}(1,2)$ and $\mathtt{b} \in \mathbb{R}^{1,2}$. The Lie algebra $\mathfrak{p}$ of the Poincaré group is the semi-direct sum $\mathbb{R}^{1,2} \oplus_{\iota} \mathfrak{o}(1,2)$ and it can be identified with the subalgebra of $\mathfrak{sl}(4,\mathbb{R})$ consisting of all 4×4 matrices of the form

$$\mathfrak{b}(\mathtt{b}, \mathtt{x}) = \begin{bmatrix} 0 & 0 \\ \mathtt{b} & \mathtt{x} \end{bmatrix},$$

where $\mathtt{b} \in \mathbb{R}^{1,2}$ and $\mathtt{x} \in \mathfrak{o}(1,2)$. For each $\mathtt{p} \in \mathbb{R}^{1,2}$, we put

$$\mathrm{j}(\mathtt{p}) = \left(1, \mathtt{p}^1, \mathtt{p}^2, \mathtt{p}^3, \frac{1}{2}(\mathtt{p}, \mathtt{p})\right) \in \mathbb{R}^{2,3}.$$

Then, $\mathrm{j}(\mathtt{p})$ is non-zero and ligth-like and the map

$$\mathbf{j} : \mathtt{p} \in \mathbb{R}^{1,2} \to |[\mathrm{j}(\mathtt{p})]| \in \mathcal{E}$$

is a conformal embedding of the Minkowski 3-space in the Einstein universe. The image of j is the open set

$$\widehat{\mathbb{R}}^{1,2} = \{|[\mathrm{V}]| \in \mathcal{E} : \langle \mathrm{V}, \mathrm{E}_4 \rangle < 0\} \subset \mathcal{E}.$$

The embedding $\mathbf{j}$ can be lifted to the faithful representation

$$\mathbf{J}:\mathtt{B}(\mathtt{b},\mathtt{X})\in \mathrm{P}^{\uparrow}_{+}\to \begin{bmatrix} 1 & 0 & 0\\ \mathtt{b} & \mathtt{X} & 0\\ \frac{1}{2}(\mathtt{b},\mathtt{b}) & \mathtt{b}^*\cdot \mathtt{X} & 1\end{bmatrix}\in \mathrm{M}^{\uparrow}_{+},$$

that intertwines the actions, i.e. $\mathbf{j}(\mathtt{B}{\cdot}\mathtt{p})=\mathbf{J}(\mathtt{B}){\cdot}\mathbf{j}(\mathtt{p})$. Note that we are identifying $\mathtt{p}$ with $(1,\mathtt{p})$. The Lie algebra representation induced by $\mathbf{J}$ is given by

$$\mathbf{J}^*:(\mathtt{b},\mathtt{x})\in \mathfrak{p}\to \begin{bmatrix} 0 & 0 & 0\\ \mathtt{b} & \mathtt{x} & 0\\ 0 & {}^*\mathtt{b} & 0\end{bmatrix}\in \mathfrak{m}, \tag{2.3}$$

where ${}^*\mathtt{p}$ is the row vector $(-\mathtt{p}^1,\mathtt{p}^2,\mathtt{p}^3)$. The images $\widehat{\mathrm{P}}^{\uparrow}_{+}\subset \mathrm{M}^{\uparrow}_{+}$ and $\widehat{\mathfrak{p}}\subset\mathfrak{m}$ of the representations $\mathbf{J}$ and $\mathbf{J}^*$ are the closed subgroup $\widehat{\mathrm{P}}^{\uparrow}_{+}=\{\mathbf{B}\in \mathrm{M}^{\uparrow}_{+}:\mathbf{B}{\cdot}\mathrm{E}_4=\mathrm{E}_4\}$ and the Lie subalgebra $\widehat{\mathfrak{p}}=\{\mathrm{b}\in\mathfrak{m}:\mathbf{b}\cdot \mathrm{E}_4)=0\}$.

Remark 2.7. Let $\mu=(\mu^i_j)$ be the Maurer-Cartan form of $\mathrm{M}^{\uparrow}_{+}$ and $\mathcal{D}$ be the left-invariant completely integrable Pfaffian differential system generated by the 1-forms μ^0_0,μ^1_4,μ^2_4 and μ^3_4. Its maximal integral submanifolds are the left cosets $\mathbf{B}\cdot\widehat{\mathrm{P}}^{\uparrow}_{+}$, $\mathbf{B}\in \mathrm{M}^{\uparrow}_{+}$. Then, if N is a connected manifold and $\Phi:\mathrm{N}\to \mathrm{M}^{\uparrow}_{+}$ is a smooth map such that

$$\Phi^*(\mu^0_0)=\Phi^*(\mu^1_4)=\Phi^*(\mu^2_4)=\Phi^*(\mu^3_4)=0,$$

there exist $\mathbf{B}\in \mathrm{M}^{\uparrow}_{+}$ and a smooth map $\widetilde{\Phi}:\mathrm{N}\to \mathrm{P}^{\uparrow}_{+}$ such that $\Phi=\mathbf{B}\cdot\mathbf{J}\circ\widetilde{\Phi}$.

Remark 2.8. Note that also the anti-de Sitter, the de-Sitter and, more generally, all 3-dimensional Robertson-Walker space-times can be conformally embedded in $\mathcal{E}$; [**16**].

Remark 2.9. Unlike in the Riemannian case, there exists a countable family of conformally flat Lorentzian manifold, not globally equivalent to each other, that admit a transitive group of conformal transformations of maximal dimension, [**13**]. For instance, beside the oriented, time-oriented conformal compactification of the Einstein universe, there are other two basic models. The first one is the un-oriented compactification $\mathcal{E}'$ which can be realized as the manifold of non-oriented isotropic lines of $\mathbb{R}^{2,3}$. Its conformal transformation group is the quotient of the pseudo-orthogonal group of $\langle -,-\rangle$ by its center $\{\pm \mathrm{Id}_{5\times 5}\}$. However, $\mathcal{E}'$ is neither orientable or time-orientable. The other basic model is "the physical Einstein universe", that is $\mathcal{E}^*=\mathbb{R}\times\mathbb{S}^2$, with the conformal structure defined by the Lorentzian pseudo-metric $-\mathrm{d}t^2+g_{\mathbb{S}^2}$. The restricted conformal group of $\mathcal{E}^*$ can be described as follows : the oriented, time-oriented compact model $\mathcal{E}$ is diffeomorphic to the Grassmannian of the oriented

2-planes of $\mathbb{R}^4$ which are Lagrangian with respect to the standard symplectic structure of $\mathbb{R}^4$ (see for instance [**10**]). The linear symplectic group[2] $\mathrm{Sp}(4,\mathbb{R})$ acts almost effectively as a group of restricted conformal transformations on $\mathcal{E}$ via its standard action on the Grassmannian of the oriented Lagrangian planes. Hence, by general facts on Lie transformation groups [**5**], the universal covering $\widehat{\mathrm{Sp}}(4,\mathbb{R})$ of $\mathrm{Sp}(4,\mathbb{R})$ acts almost effectively via conformal transformations on the left of $\mathcal{E}^*$. Then, the restricted conformal group of $\mathcal{E}^*$ is the the quotient of $\widehat{\mathrm{Sp}}(4,\mathbb{R})$ by a subgroup isomorphic to $\mathbb{Z}_2$. We refer the reader to [**25**] for a detailed analysis of the universal covering of the linear symplectic groups and to [**13**] for an explicit description of all oriented, time-oriented Lorentzian manifolds admitting a restricted conformal group of maximal dimension.

3 - Conformal Geometry of a space-like immersed surface

3.1 - *The conformal Gauss map and the quartic differential*

Let $\mathcal{S}$ be a connected surface and $f : \mathcal{S} \to \mathcal{E}$ be a smooth space-like immersion. Since $\mathcal{E}$ is oriented and time-oriented, the surface $\mathcal{S}$ can be canonically oriented by the unique future-directed time-like unit normal along f. The orientation and the conformal class of the Riemannian metric $f^*(\ell_{\mathcal{E}})$ give on $\mathcal{S}$ the structure of a Riemann surface. We shall denote by $*$ the Hodge-star operator on 1-forms of $\mathcal{S}$.

Remark 3.1. Let $\pi_2 : \mathcal{E} \cong \mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{S}^2$ be the projection of the Einstein universe onto the unit sphere of $\mathbb{V}_*^{\perp}$. If f is space-like, then $\pi_2 \circ f : \mathcal{S} \to \mathbb{S}^2$ is a local diffeomorphism. Thus, if $\mathcal{S}$ is compact $\pi_2 \circ f$ is a covering map and hence $\mathcal{S}$ is bi-holomorphic to a 2-dimensional sphere.

Definition 3.2. The *tautological bundles* of f are defined by

$$\begin{cases} \mathcal{T}_f = \{(p, \mathrm{V}) \in \mathcal{S} \times \mathbb{R}^{2,3} / |[\mathrm{V}]| = f(p)\}, \\ \mathcal{K}_f = \{(p, \mathrm{V}) \in \mathcal{S} \times \mathbb{R}^{2,3} / \mathrm{V} \in f(p)\}. \end{cases}$$

$\mathcal{T}_f$ is a principal fiber bundle with structure group $\mathbb{R}^+$. Its trivializations are the *lifts* of f, that is smooth maps $\mathrm{F} : \mathcal{S} \to \mathbb{R}^{2,3}$ into the ligth-cone of $\mathbb{R}^{2,3}$ such that $|[\mathrm{F}(p)]| = f(p)$, for every $p \in \mathcal{S}$. $\mathcal{K}_f$ is a real line bundle trivialized by lifts of f which, in turns, determine an orientation on $\mathcal{K}_f$.

Remark 3.3. Global lifts do exist for every f. This assertion can be justified as follows : consider the diffeomorphims $j_{\mathcal{E}} : \mathrm{V} \in \mathbb{S}^1 \times \mathbb{S}^2 \subset \mathbb{R}^{2,3} \to$

[2]Note that $\mathrm{Sp}(4,\mathbb{R})$ is the spin covering of $\mathrm{M}_+^{\uparrow}$.

$|[\mathrm{V}]| \in \mathcal{E}$. Then $\mathrm{F}_{\mathcal{E}} = j_{\mathcal{E}}^{-1} \circ f : \mathcal{S} \to \mathbb{R}^{2,3}$ is a lift of f, referred to as the *Einstein lift*. The lifts are defined up to a positive multiplicative factor, i.e., if F is a lift of f then any other is given by $\widetilde{\mathrm{F}} = r\mathrm{F}$, where $r : \mathcal{S} \to \mathbb{R}$ is a strictly positive smooth function.

Definition 3.4. The 4-*dimensional* (*cyclic model*) *of the AdS space* is the quadric $\mathcal{A} \subset \mathbb{R}^{2,3}$ of all unit time-like vectors of $\mathbb{R}^{2,3}$ equipped with the Lorentzian pseudo-metric $\ell_{\mathcal{A}}$ induced by the scalar product $\langle -, - \rangle$.

Remark 3.5. Let W be a point of $\mathcal{A}$. The orthogonal complement $\mathrm{W}^{\perp}$ is a 4-dimensional linear subspace of type (1,3), which coincides with the tangent space $\mathrm{T}_{\mathrm{W}}(\mathcal{A})$ of $\mathcal{A}$ at W. The set of all null-rays lying in $\mathrm{W}^{\perp}$ has two connected components

$$\begin{cases} \mathbb{S}^{\uparrow}_{\mathrm{V}} = \{|[\mathrm{V}]| \in \mathcal{E} / \langle \mathrm{V}, \mathrm{W} \rangle = 0, |[\mathrm{V} \wedge \mathrm{W}]| \in \mathcal{L}^{\uparrow}\}, \\ \mathbb{S}^{\downarrow}_{\mathrm{V}} = \{|[\mathrm{V}]| \in \mathcal{E} / \langle \mathrm{V}, \mathrm{W} \rangle = 0, |[\mathrm{V} \wedge \mathrm{W}]| \in \mathcal{L}^{\downarrow}\}. \end{cases}$$

Each of them is a 2-dimensional *"round" sphere* of the Einstein universe. $\mathbb{S}^{\uparrow}_{\mathrm{V}}$ and $\mathbb{S}^{\downarrow}_{\mathrm{V}}$ are totally umbilical and space-like. Moreover, the image of any totally umbilical space-like immersion is an open subset of exactly one round 2-sphere. This gives a geometrical interpretation of the 4-dimensional AdS space as the manifold of all round 2-spheres of $\mathcal{E}$.

Definition 3.6. A *conformal unit normal* along f is a time-like unit vector field $\mathcal{N} : \mathcal{S} \to \mathcal{A}$ such that

$$|[\mathrm{F}|_p \wedge \mathcal{N}|_p]| \in \mathcal{L}^{\uparrow}, \quad \langle \mathrm{F}|_p, \mathcal{N}|_p \rangle = 0, \quad \langle \mathrm{dF}|_p, \mathcal{N}|_p \rangle = 0, \quad \forall p \in S.$$

If F is a lift and $\mathcal{N}$ is a unit normal then $\langle \mathrm{dF}, \mathrm{dF} \rangle$ and $\langle \mathrm{dF}, \mathrm{d}\mathcal{N} \rangle$ are symmetric quadratic forms on $\mathcal{S}$. Let

$$\langle \mathrm{dF}, \mathrm{d}\mathcal{N} \rangle = \langle \mathrm{dF}, \mathrm{d}\mathcal{N} \rangle^{(2,0)} + \langle \mathrm{dF}, \mathrm{d}\mathcal{N} \rangle^{(1,1)} + \langle \mathrm{dF}, \mathrm{d}\mathcal{N} \rangle^{(0,2)}$$

be the decomposition of $\langle \mathrm{dF}, \mathrm{d}\mathcal{N} \rangle$ into bidegrees.

Lemma 3.7. *There exists a unique conformal unit normal along f, denoted by $\mathcal{N}_f$, such that*

$$\langle \mathrm{dF}, \mathrm{d}\mathcal{N}_f \rangle^{(1,1)} = 0.$$

Proof. First we prove that conformal unit normals do exist along any space-like immersion. Let us consider the Einstein lift $\mathrm{F}_{\mathcal{E}}$ of f. In the present context, we think of $\mathcal{E}$ as $\mathbb{S}^1 \times \mathbb{S}^2$. Denote by $\mathrm{F}'_{\mathcal{E}}$ and $\mathrm{F}''_{\mathcal{E}}$ the components of $\mathrm{F}_{\mathcal{E}}$ with respect to the splitting $\mathbb{V}_* \oplus \mathbb{V}_*^{\perp}$ of $\mathbb{R}^{2,3}$. For each $p \in \mathcal{S}$, the normal bundle of $\mathcal{E} \subset \mathbb{R}^{2,3}$ at $\mathrm{F}_{\mathcal{E}}(p)$ is spanned by $\mathrm{F}'_{\mathcal{E}}|_p$ and $\mathrm{F}''_{\mathcal{E}}|_p$. Choose a local chart $(u,v) : U \to \mathbb{R}^2$ on an open neighborhood U of the point p. Then, $\mathrm{F}'_{\mathcal{E}}|_q, \mathrm{F}''_{\mathcal{E}}|_q, \partial_u \mathrm{F}_{\mathcal{E}}|_q$ and $\partial_v \mathrm{F}_{\mathcal{E}}|_q$ are linearly independent and span a 4-dimensional vector sub-space $\mathbb{W}|_q \subset \mathbb{R}^{2,3}$ of signature $(1,3)$, for every $q \in U$. Thus, there exists a unique time-like unit vector field $\widetilde{\mathcal{N}} : U \to \mathbb{R}^{2,3}$ such that $\widetilde{\mathcal{N}}|_q \in \mathbb{W}|_q^{\perp}$ and that $|[\mathrm{F}_{\mathcal{E}}|_q \wedge \widetilde{\mathcal{N}}|_q]|$ is a future-directed causal plane of type $(0,-1)$, for every $q \in U$. By construction, $\widetilde{\mathcal{N}}$ is a conformal unit normal along f and its definition is independent on the choice of the chart. Thus, there exists a unique global conformal unit normal $\mathcal{N}_{\mathcal{E}}$ such that $\mathcal{N}_{\mathcal{E}}|_U = \widetilde{\mathcal{N}}$.

Let F be a lift and $\mathcal{N}$ be a conformal unit normal. Since $\langle \mathrm{dF}, \mathrm{d}\mathcal{N} \rangle$ is real, its $(0,2)$ component is the complex conjugate of the $(2,0)$ component and $\langle \mathrm{dF}, \mathrm{d}\mathcal{N} \rangle^{(1,1)}$ is a multiple of $\langle \mathrm{dF}, \mathrm{dF} \rangle$. Hence there exists a smooth real-valued function $\sigma : \mathcal{S} \to \mathbb{R}$ such that $\langle \mathrm{dF}, \mathrm{d}\mathcal{N} \rangle^{(1,1)} = \sigma \langle \mathrm{dF}, \mathrm{dF} \rangle$. Let $\widetilde{\mathcal{N}}$ be another unit normal. Then, $\widetilde{\mathcal{N}} = \mathcal{N} + s\mathrm{F}$, where s is a smooth function. This implies

$$\langle \mathrm{d}\widetilde{\mathrm{F}}, \mathrm{d}\widetilde{\mathcal{N}} \rangle^{(1,1)} = \langle \mathrm{dF}, \mathrm{d}\mathcal{N} \rangle^{(1,1)} + s \langle \mathrm{dF}, \mathrm{dF} \rangle.$$

Putting $s = -\sigma$, we have $\langle \mathrm{d}\widetilde{\mathrm{F}}, \mathrm{d}\widetilde{\mathcal{N}} \rangle^{(1,1)} = 0$. We have thus proved the result. □

Definition 3.8. The unique unit normal $\mathcal{N}_f$ such that $\langle \mathrm{dF}, \mathrm{d}\mathcal{N}_f \rangle^{(1,1)} = 0$ is said to be the *conformal Gauss map* of f. The symmetric differential form $\mathcal{B}_f := \langle \mathrm{d}\mathcal{N}_f, \mathrm{d}\mathcal{N}_f \rangle$ and the exterior differential 2-form $\beta_f = \mathrm{d}^{\mathrm{t}}\mathcal{N}_f \cdot \mathtt{m} \wedge *\mathrm{d}\mathcal{N}_f$ depend only on the immersion f. They are called the *Blaschke line element* and the *Blaschke area element* of f respectively.

Definition 3.9. Let f be a space-like immersion with conformal Gauss map $\mathcal{N}_f$ and let $z : U \to \mathbb{C}$ be a holomorphic chart. The quartic differential $\widetilde{\mathcal{Q}} = \langle \partial^2_{zz}\mathcal{N}_f, \partial^2_{zz}\mathcal{N}_f \rangle dz^4$ doesn't depend on the choice of the chart. Then, there exists a global cross section $\mathcal{Q}_f \in \Omega^{(4,0)}(\mathcal{S})$, the *Bryant's quartic differential*, such that $\mathcal{Q}_f|_U = \widetilde{\mathcal{Q}}$.

Remark 3.10. The quartic differential $\mathcal{Q}_f$ was introduced by R. Bryant in [**6**, **7**] as a basic tool in analyzing Willmore immersions in the conformal 3-sphere. His definition may seem different from ours. We have followed the definition given by Eschenburg, [**12**], in a manuscript which, to our knowledge, has never been published.

Definition 3.11. Let $f : \mathcal{S} \to \mathcal{E}$ be a space-like immersion and $\beta_f \in \Omega^2(\mathcal{S})$ be the Blaschke area element of f. For every compact domain $\mathrm{K} \subset \mathcal{S}$, the *Blaschke energy* of f on K is the integral

$$\mathfrak{B}_{f,\mathrm{K}} = \int_{\mathrm{K}} \beta_f.$$

The critical points of the Blaschke energy functional with respect to compactly supported variations are called *Willmore or M-maximal immersions.*

The following result is due to Alías and Palmer [**1**] :

Theorem 3.12. *$f : \mathcal{S} \to \mathcal{E}$ is a Willmore space-like immersion if and only if its conformal Gauss map $\mathcal{N}_f : \mathcal{S} \to \mathcal{A}$ is a conformal harmonic map.*

Remark 3.13. The previous Theorem is the Lorentzian counterpart of a classical result of W. Blaschke and G. Thomsen, [**4**, **26**], which characterises Willmore immersions in the 3-sphere via the conformality and harmonicity of the conformal Gauss map (see also [**6**, **7**, **17**]).

Remark 3.14. The Blaschke area element and the Blaschke energy can be easily generalized for space-like immersions into any 3-dimensional Lorentzian conformal space : let $[\ell]$ be a conformal class of Lorentzian pseudo-metrics on a 3-dimensional manifold M and let $f : \mathcal{S} \to M$ be a space-like immersion of a connected, oriented surface $\mathcal{S}$ into M. Fix and choose a Lorentzian metric ℓ in the conformal class, denote by $\mathfrak{A}_f : \mathrm{T}(\mathcal{S}) \to \mathrm{T}(\mathcal{S})$ the trace-free part of the shape operator of f with respect to ℓ and by dA the area form of $f^*(\ell)$. Then, the exterior-differential 2-form $\beta_f = \det(\mathfrak{A}_f)\mathrm{dA}$ doesn't depend on the choice of ℓ in the conformal class. If the target is the Einstein universe, β_f gives back the Blaschke area element. We say that f is Willmore (or M-maximal) if f is a critical point of the action functional defined by the integral of the 2-form β_f with respect to compactly supported variations. Note that if $\pi : (M', [\ell']) \to (M, [\ell])$ is a conformal covering map, then $f : \mathcal{S} \to M'$ is a Willmore immersion if and only if $\pi \circ f : \mathcal{S} \to M$ is a Willmore immersion. However, in the general case the space of totally umbilical space-like surfaces is not anymore a manifold and one can't characterize Willmore immersions via the harmonicity of a suitable Gauss map.

3.2 - *Second-order conformal frames*

Definition 3.15. A *second-order frame* along f is a smooth map

$$\mathbf{A} = (\mathrm{A}_0, ..., \mathrm{A}_4) : U \to \mathrm{M}^{\uparrow}_{+}$$

defined on an open neighborhood of $\mathcal{S}$ such that $|[\mathrm{A}_0]| = f|_U$ and $\mathrm{A}_1 = \mathcal{N}_f|_U$.

Lemma 3.16. *Second-order frames do exist near any point of $\mathcal{S}$.*

Proof. Let p_* be a point of $\mathcal{S}$. Choose $\mathbf{C} \in \mathrm{M}^{\uparrow}_{+}$ such that $\mathbf{C} \cdot f(p_*) \in \widehat{\mathbb{R}}^{1,2}$. Let $U \subset \mathcal{S}$ be a simply connected open coordinate neighborhood of p_* such that $U \subset \widehat{\mathbb{R}}^{1,2}$. Then, there exists a unique space-like immersion $\mathtt{f} : U \to \mathbb{R}^{1,2}$ such that $\mathbf{j} \circ \mathtt{f} = \mathbf{C} \cdot f$. Let $\mathbf{n}_{\mathtt{f}}$ be the Lorentzian Gauss map of $\mathtt{f}$, that is the unique future-directed unit time-like normal along $\mathtt{f}$. Since U is a coordinate neighborhood, there exist two unit tangent vector fields $\mathtt{B}_2, \mathtt{B}_3$ along $\mathtt{f}$ such that $(\mathbf{n}|_p, \mathtt{B}_2|_p, \mathtt{B}_3|_p)$ is a pseudo-orthogonal basis of $\mathbb{R}^{1,2}$, for every $p \in U$. The smooth map

$$\mathtt{B} = (\mathtt{f}, \mathbf{n}_{\mathtt{f}}, \mathtt{B}_2, \mathtt{B}_3) : U \to \mathrm{P}^{\uparrow}_{+}$$

is a *first-order Lorentzian frame* along $\mathtt{f}$, i.e. a lift of $\mathtt{f}$ to $\mathrm{P}^{\uparrow}_{+}$ such that

$$\mathtt{B}^{-1}\mathrm{d}\mathtt{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \beta^2_1 & \beta^3_1 \\ \beta^2_0 & \beta^2_1 & 0 & -\beta^3_2 \\ \beta^3_0 & \beta^3_1 & \beta^3_2 & 0 \end{bmatrix},$$

where

- $\zeta_{\mathtt{B}} = \beta^2_0 + i\beta^3_0$ is a 1-form of type $(1,0)$, non-zero at every point of U;
- $\beta^2_1 - i\beta^3_1 = \mathrm{h}_{\mathtt{B}}\zeta_{\mathtt{B}} + \dfrac{i}{2}\mathrm{H}\overline{\zeta}_{\mathtt{B}}$, where H is the mean curvature of $\mathtt{f}$.

Next, consider the map $\widetilde{\mathbf{A}} = \mathbf{C}^{-1} \cdot \mathbf{J} \circ \mathtt{B} : U \to \mathrm{M}^{\uparrow}_{+}$. Then, $|[\widetilde{\mathbf{A}_0}]| = f|_U$ and $\widetilde{\mathbf{A}}^{-1}\mathrm{d}\widetilde{\mathbf{A}} = \mathbf{J}^*(\beta)$. Let $\mathbf{X} : U \to \mathrm{M}^{\uparrow}_{+}$ be defined by

$$\mathbf{X} = \begin{bmatrix} 1 & -\mathrm{H}/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \mathrm{H}/2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We put $\mathbf{A} = \widetilde{\mathbf{A}} \cdot \mathbf{X}$. Then, $|[\mathrm{A}_0]| = f|_U$ and

$$\alpha = \mathbf{A}^{-1}\mathrm{d}\mathbf{A} = \mathbf{X}^{-1}(\widetilde{\mathbf{A}}^{-1}\mathrm{d}\widetilde{\mathbf{A}}\mathbf{X} + \mathrm{d}\mathbf{X}).$$

From this we infer that

$$\alpha^1_0 = 0, \quad \alpha^2_0 + i\alpha^3_0 = \zeta_{\mathtt{B}}, \quad \alpha^2_1 - i\alpha^3_1 = \mathrm{h}_{\mathtt{B}}\zeta_{\mathtt{B}}.$$

This implies that $\mathbf{A}$ is a second-order frame field along f. $\square$

Remark 3.17. From the proof of the previous Proposition it follows that if $\mathtt{f}$ is a maximal space-like immersion in $\mathbb{R}^{1,2}$, then the conformal Gauss map of $f = \mathbf{j} \circ \mathtt{f}$ can be written as

$$\mathcal{N}_f =^t (0, \mathtt{n_f}, \mathtt{f}^* \cdot \mathtt{n_f}). \tag{3.1}$$

Given $r > 0$, $x, y \in \mathbb{R}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ we put

$$\mathbf{B}(r,\theta,x,y) = \begin{bmatrix} r & 0 & x & y & \frac{1}{2r}(x^2+y^2) \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) & \frac{1}{r}(\cos(\theta)x - \sin(\theta)y) \\ 0 & 0 & \sin(\theta) & \cos(\theta) & \frac{1}{r}(\sin(\theta)x + \cos(\theta)y) \\ 0 & 0 & 0 & 0 & r^{-1} \end{bmatrix} \in \mathrm{M}_+^{\uparrow}. \tag{3.2}$$

Let K_2 the closed subgoup of $\mathrm{M}_+^{\uparrow}$ defined by

$$\mathrm{K}_2 = \{\mathbf{B}(r,x,y,\theta) : r > 0, x, y, \in \mathbb{R}, \theta \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

If $\mathbf{A}$ and $\mathbf{A}'$ are second order frames on U, then there exists a unique map $\mathbf{B} : U \to \mathrm{K}_2$ such that $\mathbf{A}' = \mathbf{A} \cdot \mathbf{B}$. For each $p \in \mathcal{S}$ we denote by $\mathcal{F}_2|_p$ the set of all $\mathbf{X} \in \mathrm{M}_+^{\uparrow}$ such that $\mathbf{X} = \mathbf{A}|_p$, where $\mathbf{A} : U \to \mathrm{M}_+^{\uparrow}$ is a second order frame along f defined on an open neighborhood of p. Then

$$\mathcal{F}_2 = \{(p, \mathbf{X}) \in \mathcal{S} \times \mathrm{M}_+^{\uparrow} : \mathbf{X} \in \mathcal{F}_2|_p\}$$

is a principal fiber bundle over $\mathcal{S}$ with structural group K_2. We let $\mathcal{T}_2 : \mathcal{F}_2 \to \mathrm{M}_+^{\uparrow}$ be the map such that $\mathcal{T}_2(p, \mathbf{X}) = \mathbf{X}$, for every $(p, \mathbf{X}) \in \mathcal{F}_2$. The second-order frames are the local trivializations of $\mathcal{F}_2$. Let $\mathbf{A} : U \to \mathrm{M}_+^{\uparrow}$ be a second-order frame and $\alpha = \mathbf{A}^{-1}\mathrm{d}\mathbf{A}$ be the pull-back of the Maurer-Cartan form of $\mathrm{M}_+^{\uparrow}$. This is an $\mathfrak{m}$-valued exterior differential 1-form on U. Its entries are denoted by α^i_j, $0 \le i, j \le 4$. The identities $\langle \mathrm{dA}_0, \mathrm{A}_1\rangle = \langle \mathrm{dA}_0, \mathrm{A}_2\rangle = 0$ imply that

$$\alpha = \begin{bmatrix} \alpha^0_0 & -\alpha^1_4 & \alpha^2_4 & \alpha^3_4 & 0 \\ 0 & 0 & \alpha^2_1 & \alpha^3_1 & \alpha^1_4 \\ \alpha^2_0 & \alpha^2_1 & 0 & -\alpha^3_2 & \alpha^2_4 \\ \alpha^3_0 & \alpha^3_1 & \alpha^3_2 & 0 & \alpha^3_4 \\ 0 & 0 & \alpha^2_0 & \alpha^3_0 & -\alpha^0_0 \end{bmatrix}, \tag{3.3}$$

where $\zeta_{\mathbf{A}} := \alpha_0^2 + i\alpha_0^3$ is a complex 1-form of type $(1,0)$, non-zero at each point of U. The first column vector of $\mathbf{A}$ is a lift of f and its second column vector is the restriction on U of the conformal Gauss map of f, so that $\langle \mathrm{dA}_0, \mathrm{d}\mathcal{N}\rangle^{(1,1)} = 0$. Hence, $\eta_{\mathbf{A}} = \alpha_1^2 - i\alpha_1^3$ is a 1-form of type $(1,0)$. Consequently we may write

$$\eta_{\mathbf{A}} = \mathrm{h}_{\mathbf{A}}\zeta_{\mathbf{A}},$$

where $\mathrm{h}_{\mathbf{A}}$ is a complex valued function. If $\mathbf{A}'$ is any other second order frame on U, then $\mathbf{A}' = \mathbf{A}\cdot\mathbf{B}$, where $\mathbf{B} = \mathbf{B}(r,\theta,x,y) : U \to \mathrm{K}_2$ is a smooth map into the structure group. The 1-forms α and α' are related by

$$\alpha' = \mathbf{B}^{-1}\alpha\mathbf{B} + \mathbf{B}^{-1}\mathrm{d}\mathbf{B}. \tag{3.4}$$

This implies

$$\zeta_{\mathbf{A}'} = re^{-i\theta}\zeta_{\mathbf{A}}, \quad \eta_{\mathbf{A}'} = e^{i\theta}\eta_{\mathbf{A}}, \tag{3.5}$$

and

$$\begin{cases} \alpha_0'^0 = \alpha_0^0 + \dfrac{dr}{r} - (x\cos(\theta) - y\sin(\theta))\alpha_0^2 - (x\sin(\theta) + y\cos(\theta))\alpha_0^3, \\ \alpha_4'^1 = \dfrac{1}{r}(\alpha_4^1 + (x\cos(\theta) - y\sin(\theta))\alpha_1^2 + (x\sin(\theta) + y\cos(\theta))\alpha_1^3). \end{cases} \tag{3.6}$$

Definition 3.18. Let (U,z) be a complex chart[3]. A second-order frame $\mathbf{A} : U \to \mathrm{M}_+^{\uparrow}$ is said to be adapted to (U,z) if $\zeta_{\mathbf{A}} = \mathrm{d}z$ and $\alpha_0^0 = 0$.

Lemma 3.19. *If (U,z) is a complex chart then there exists a unique second-order frame adapted to (U,z).*

Proof. The uniqueness is a straightforward consequence of the transformation laws (3.5) and (3.6). Arguing as in the proof of Lemma 3.16 we may conclude that there exists a second-order frame $\mathbf{A}$ defined on the coordinate neighborhood U. Then, $\alpha_0^2 + i\alpha_0^3 = re^{i\theta}\mathrm{d}z$ where r is a positive smooth function and $\theta : U \to \mathbb{R}/2\pi\mathbb{Z}$ is a smooth map. Putting $\mathbf{A}' = \mathbf{A}\cdot\mathbf{B}(r^{-1},\theta,0,0)$ we get a second-order frame such that $\eta_{\mathbf{A}'} = \mathrm{h}_{\mathbf{A}'}dz$. We write $\alpha_0'^0 = a\mathrm{d}x + b\mathrm{d}y$, where $a,b : U \to \mathbb{R}$ are smooth functions. Let $\mathbf{A}''$ be the second-order frame defined by $\mathbf{A}'' = \mathbf{A}'\mathbf{B}(0,0,a,b)$. Then, $\mathbf{A}''$ is a second-order frame adapted to (U,z). $\square$

Let $\mathfrak{O} = \{(U_a, z_a)\}_{a\in J}$ be the atlas of all holomorphic charts of $\mathcal{S}$ and $\mathbf{A}_a = (\mathrm{A}_0^a, ..., \mathrm{A}_4^a)$ be the second-order frame adapted to (U_a, z_a). For each $a \in J$ there exist smooth functions $\mathrm{h}_a, \ell_a, \mathrm{k}_a : U \to \mathbb{C}$ such that

$$\alpha_4^1 = \ell_a\mathrm{d}z_a + \overline{\ell}_a\mathrm{d}\overline{z}_a, \quad \alpha_1^2 - i\alpha_1^3 = \mathrm{h}_a\mathrm{d}z_a, \quad \alpha_4^2 + i\alpha_4^3 = \frac{1}{2}|\mathrm{h}_a|^2\mathrm{d}z_a + \mathrm{k}_a\mathrm{d}\overline{z}_a. \tag{3.7}$$

[3] We implicitly assume that the coordinate neighborhoods of complex charts are connected.

The last equation in (3.7) can be proved as follows : differentiating $\alpha_0^0 = 0$ and using the structure equations $\mathrm{d}\alpha + \alpha \wedge \alpha = 0$ we get $\alpha_4^2 \wedge \mathrm{d}x_a + \alpha_4^3 \wedge \mathrm{d}y_a = 0$. This implies the existence of smooth functions $\mathrm{k}_a : U_a \to \mathbb{C}$ and $\mathrm{c}_a : U \to \mathbb{R}$ such that $\alpha_4^2 + i\alpha_4^3 = \mathrm{c}_a \mathrm{d}z_a + \mathrm{k}_a \mathrm{d}\overline{z}_a$. Differentiating $\alpha_2^3 = 0$ and using again the structure equations we find $\mathrm{d}y_a \wedge \alpha_4^2 + \alpha_1^3 \wedge \alpha_1^2 + \alpha_4^3 \wedge \mathrm{d}x_a = 0$. Taking into account that $\alpha_1^2 - i\alpha_1^3 = \mathrm{h}_a \mathrm{d}z_a$, the previous equation implies that $\mathrm{c}_a = |\mathrm{h}_a|^2/2$. The remaining equations originated from the Maurer-Cartan equations of the group $\mathrm{M}_+^{\uparrow}$ are

$$\begin{cases} \mathrm{d}\alpha_4^1 = \alpha_4^2 \wedge \alpha_1^2 + \alpha_4^3 \wedge \alpha_1^3, \\ \mathrm{d}(\alpha_1^2 - i\alpha_1^3) = -\alpha_4^1 \wedge (\alpha_0^2 - i\alpha_0^3), \\ \mathrm{d}(\alpha_4^2 + i\alpha_4^3) = \alpha_4^1 \wedge (\alpha_2^2 + i\alpha_1^3). \end{cases}$$

Using (3.7) these equation can be rewritten as follows

$$\partial_{\overline{z}_a}\mathrm{h}_a = \ell_a, \quad \partial_{z_a}\mathrm{k}_a = \frac{1}{2}\partial_{\overline{z}_a}(|\mathrm{h}_a|^2) + \overline{\mathrm{h}}_a\ell_a, \quad \mathrm{Im}(\partial_{\overline{z}_a}\ell_a) = \frac{1}{2}\mathrm{Im}(\mathrm{h}_a\mathrm{k}_a). \tag{3.8}$$

Vice versa, let (U_a, z_a) be a complex chart on a simply-connected coordinate neighborhood and $\mathrm{h}_a, \ell_a, \mathrm{k}_a$ be complex-valued smooth functions satisfying (3.8). If we put $\alpha_0^2 + i\alpha_0^3 = \mathrm{d}z_a$, $\alpha_0^0 = 0$, if we define α_4^1, $\alpha_1^2, \alpha_1^3, \alpha_4^2$ and α_4^3 as in (3.7) and if α is as in (3.3) then α is an $\mathfrak{m}$-valued 1-form satisfying the Maurer-Cartan equations $\mathrm{d}\alpha = -\alpha \wedge \alpha$. As a consequence of the Frobenius theorem (see for instance [**17, 28**]) the 1-form α can be integrated to a smooth map $\mathbf{A}_a : U_a \to \mathrm{M}_+^{\uparrow}$ such that $f_a : p \in U_a \to |[\mathrm{A}_0^a]| \in \mathcal{E}$ is a space-like immersion. Furthermore, $\mathbf{A}_a$ is the second-order frame along f_a adapted to (U_a, z_a).

3.3 - *The Poynting field and the dual map of a space-like immersion*

Put

$$\delta(J) = \{(a, b) \in J \times J : U_a \cap U_b \neq \emptyset\}.$$

For each $(a, b) \in \delta(J)$ we can write

$$\mathrm{d}z_b = \mathrm{R}_b^a \mathrm{d}z_a = r_b^a e^{i\phi_b^a} \mathrm{d}z_a, \tag{3.9}$$

where $\mathrm{R}_b^a \in \mathcal{O}(U_a \cap U_b)$, $r_b^a = |\mathrm{R}_b^a|$, and $\phi_b^a : U_a \cap U_b \to \mathbb{R}/2\pi\mathbb{Z}$ is a smooth function. We then have

$$\mathbf{A}_b = \mathbf{A}_a \cdot \mathbf{B}(r_b^a, -\phi_b^a, x_b^a, y_b^a), \tag{3.10}$$

where $v_b^a = x_b^a + iy_b^a : U_a \cap U_b \to \mathbb{C}$ is a complex-valued smooth function. We can rewrite the transformation rule (3.10) as follows :

(3.11)
$$\begin{cases} \mathrm{A}_0^b = r_b^a \mathrm{A}_0^a, \\ \mathrm{A}_1^b = \mathrm{A}_1^a = \mathcal{N}|_{U_a \cap U_b}, \\ \mathrm{A}_2^b - i\mathrm{A}_3^b = \overline{v}_b^a \mathrm{A}_0^a + e^{-i\phi_b^a}(\mathrm{A}_2^a - i\mathrm{A}_3^a), \\ \mathrm{A}_4^b = \dfrac{|v_b^a|^2}{2r_b^a}\mathrm{A}_0^a + \dfrac{v_b^a}{2r_b^a}e^{-i\phi_b^a}(\mathrm{A}_2^a - i\mathrm{A}_3^a) + \dfrac{\overline{v}_b^a}{2r_b^a}e^{i\phi_b^a}(\mathrm{A}_2^a + i\mathrm{A}_3^a) + \dfrac{1}{r_b^a}\mathrm{A}_4^a. \end{cases}$$

From (3.5) and (3.6) we obtain

$$\mathrm{h}_b = \frac{1}{r_b^a}e^{-2i\phi_b^a}\mathrm{h}_a, \quad \ell_b = \frac{e^{-i\phi_b^a}}{(r_b^a)^2}\ell_a + \frac{e^{-2i\phi_b^a}v_b^a}{2(r_b^a)^2}\mathrm{h}_a. \tag{3.12}$$

For each $a \in J$ we put

$$\mathrm{Y}_a = 2|\ell_a|^2\mathrm{A}_0^a - \ell_a\overline{\mathrm{h}}_a(\mathrm{A}_2^a - i\mathrm{A}_3^a) - \overline{\ell}_a\mathrm{h}_a(\mathrm{A}_2^a + i\mathrm{A}_3^a) + |\mathrm{h}_a|^2\mathrm{A}_4^a. \tag{3.13}$$

This is a differentiable map from U_a to the light cone of $\mathbb{R}^{2,3}$. In addition, using (3.11) and (3.12) it is a computational matter to check that

$$\mathrm{Y}_b|_{U_a \cap U_b} = (r_b^a)^{-3}\mathrm{Y}_a|_{U_a \cap U_b}, \tag{3.14}$$

for every $(a,b) \in \delta(J)$.

Definition 3.20. From (3.14) and (3.11) if follows that there exists a global cross section Y_f, of the vector bundle[4] $\mathcal{K}_f^3 \otimes (\mathcal{S} \times \mathbb{R}^{2,3})$ such that $\mathrm{Y}_f|_{U_a} = (\mathrm{A}_0^a)^3 \otimes \mathrm{Y}_a$. We call Y_f the *Poynting field* of f.

Definition 3.21. Let $\mathcal{Z}$ be the zero locus of Y_f and $\mathcal{S}_* = \mathcal{S} - \mathcal{Z}$. Then there exists a unique smooth map $\widehat{f} : \mathcal{S}_* \to \mathcal{E}$ such that $\widehat{f}|_{U_a} = |[\mathrm{Y}_a]|_{\mathcal{S}_*}$, for every $a \in J$ such that $U_a \cap \mathcal{Z} = \emptyset$. The map $\widehat{f} : \mathcal{S}_* \to \mathcal{E}$ is called the *dual map* of the space-like immersion f.

Remark 3.22. Our construction is in analogy with the one given by R. Bryant, [**6**], in the context of Möbius geometry.

[4]We use the notation E^n for the n-th tensor power of a vector bundle $E \to \mathcal{S}$ and E^{-n} for the n-th tensor power of the dual bundle E^*.

4 - Proof of Theorem A and Theorem B

Theorem A. *Let $\mathcal{S}$ be a compact 2-dimensional connected manifold and $f : \mathcal{S} \to \mathcal{E}$ be a space-like Willmore immersion. Then $f(\mathcal{S})$ is a totally umbilical round 2-sphere.*

Proof. The proof is organized into four Lemmas, two Corollaries and a concluding reasoning. At some point we use a consequence of a well known fact about complex vector bundles on a Riemann surface. The proof of the Property is given in the Appendix.

Lemma 4.1. *Let $f : \mathcal{S} \to \mathcal{E}$ be a space-like Willmore immersion. Then for every holomorphic chart (U_a, z_a) the following identity holds true*

$$\partial_{\bar{z}_a}\ell_a = \frac{1}{2}\mathrm{h}_a\mathrm{k}_a. \tag{4.1}$$

Proof. As a first step we compute the tension field of the conformal Gauss map. We choose and fix a holomorphic chart (U_a, z_a). We put

$$\mathrm{h}_a = \mathrm{h}^2_2 - i\mathrm{h}^2_3, \quad \ell_a = \frac{1}{2}(\ell_2 - i\ell_3), \quad \mathrm{k}_a = \frac{1}{2}((\mathrm{k}^2_2 - \mathrm{k}^3_3) + 2i\mathrm{k}^2_3), \quad \mathrm{k}^2_2 + \mathrm{k}^3_3 = |\mathrm{h}_a|^2.$$

Then, the relevant entries of the 1-form $\alpha = \mathbf{A}_a^{-1}\mathrm{d}\mathbf{A}_a$ can be written as

$$\begin{cases} \alpha^2_0 + i\alpha^3_0 = dz_a, & \alpha^4_1 = \ell_2\mathrm{d}x_a + \ell_3\mathrm{d}y_a, \\ \alpha^2_1 = \mathrm{h}^2_2\mathrm{d}x_a + \mathrm{h}^2_3\mathrm{d}y_a, & \alpha^3_1 = \mathrm{h}^3_2\mathrm{d}x_a - \mathrm{h}^2_2\mathrm{d}y_a, \\ \alpha^2_4 = \mathrm{k}^2_2\mathrm{d}x_a + \mathrm{k}^2_3\mathrm{d}y_a, & \alpha^3_4 = \mathrm{k}^2_3\mathrm{d}x_a + \mathrm{k}^3_3\mathrm{d}y_a. \end{cases}$$

The normal space of the AdS space $\mathcal{A}$ at $\mathcal{N}_f|_p$ is spanned by $\mathrm{A}^a_1|_p$, for every $p \in U_a$. Hence, $(\mathrm{A}^a_0, \mathrm{A}^a_2, \mathrm{A}^a_3, \mathrm{A}^a_4)$ is a local trivialization of the bundle $\mathcal{N}^*_f(\mathrm{T}(\mathcal{A}))$ and the pull-back of the Levi-Civita covariant derivative of $\mathcal{A}$, denoted by D, acts as follows :

$$\begin{cases} D\mathrm{A}^a_0 = dx_a\mathrm{A}^a_2 + \mathrm{d}y_a\mathrm{A}^a_3, \\ D\mathrm{A}^a_2 = \alpha^2_4\mathrm{A}^a_0 + \mathrm{d}x_a\mathrm{A}^a_4 = (\mathrm{k}^2_2\mathrm{d}x_a + \mathrm{k}^2_3\mathrm{d}y_a)\mathrm{A}^a_0 + \mathrm{d}x_a\mathrm{A}^a_4, \\ D\mathrm{A}^a_3 = \alpha^3_4\mathrm{A}^a_0 + \mathrm{d}y_a\mathrm{A}^a_4 = (\mathrm{k}^2_3\mathrm{d}x_a + \mathrm{k}^3_3\mathrm{d}y_a)\mathrm{A}^a_0 + \mathrm{d}y_a\mathrm{A}^a_4, \\ D\mathrm{A}^a_4 = \alpha^2_4\mathrm{A}^a_2 + \alpha^3_4\mathrm{A}^a_3 = (\mathrm{k}^2_2\mathrm{d}x_a + \mathrm{k}^2_3\mathrm{d}y_a)\mathrm{A}^a_2 + (\mathrm{k}^2_3\mathrm{d}x_a + \mathrm{k}^3_3\mathrm{d}y_a)\mathrm{A}^a_3. \end{cases}$$

On $\mathcal{S}$ we choose the Levi-Civita covariant derivative ∇ of the flat metric $g_a = (\mathrm{d}x_a)^2 + (\mathrm{d}y_a)^2$ so that $\nabla\mathrm{d}x_a = \nabla\mathrm{d}y_a = 0$. Let $\widetilde{D}$ be the covariant derivative

$\nabla \otimes D$ on $\mathrm{T}^*(\mathcal{S}) \otimes \mathcal{N}_f^*(\mathrm{T}(\mathcal{A}))$. Then,

$$\begin{aligned}\widetilde{D}(\mathrm{d}\mathcal{N}_f)) = \widetilde{D}\left(-(\ell_2 \mathrm{d}x_a + \ell_3 \mathrm{d}y_a)\mathrm{A}_0^a + (\mathrm{h}_2^2 \mathrm{d}x_a + \mathrm{h}_3^2 \mathrm{d}y_a)\mathrm{A}_2^a + (\mathrm{h}_3^2 \mathrm{d}x_a - \mathrm{h}_2^2 \mathrm{d}y_a)\mathrm{A}_3^a\right) \\ = \Phi^0 \mathrm{A}_0^a + \Phi^2 \mathrm{A}_2^a + \Phi^3 \mathrm{A}_3^a + \Phi^4 \mathrm{A}_4^a,\end{aligned}$$

where Φ^j, $j + 0, 2, 3, 4$ are the bilinear forms on U_a given by

$$\begin{cases} \Phi^0 = -(\mathrm{d}\ell_2 \otimes \mathrm{d}x_a + \mathrm{d}\ell_3 \otimes \mathrm{d}y_a) + (\mathrm{h}_2^2 \mathrm{d}x_a + \mathrm{h}_3^2 \mathrm{d}y_a) \otimes (\mathrm{k}_2^2 \mathrm{d}x_a + \mathrm{k}_3^2 \mathrm{d}y_a) \\ \qquad + (\mathrm{h}_3^2 \mathrm{d}x_a - \mathrm{h}_2^2 \mathrm{d}y_a) \otimes (\mathrm{k}_3^2 \mathrm{d}x_a + \mathrm{k}_3^3 \mathrm{d}y_a), \\ \Phi^2 = \mathrm{d}\mathrm{h}_2^2 \otimes \mathrm{d}x_a + \mathrm{d}\mathrm{h}_3^2 \otimes \mathrm{d}y_a - (\ell_2 \mathrm{d}x_a + \ell_3 \mathrm{d}y_a) \otimes \mathrm{d}x_a, \\ \Phi^3 = \mathrm{d}\mathrm{h}_3^2 \otimes \mathrm{d}x_a - \mathrm{d}\mathrm{h}_2^2 \otimes \mathrm{d}y_a - (\ell_2 \mathrm{d}x_a + \ell_3 \mathrm{d}y_a) \otimes \mathrm{d}y_a, \\ \Phi^4 = (\mathrm{h}_2^2 \mathrm{d}x_a + \mathrm{h}_3^2 \mathrm{d}y_a) \otimes \mathrm{d}x_a + (\mathrm{h}_3^2 \mathrm{d}x_a - \mathrm{h}_2^2 \mathrm{d}y_a) \otimes \mathrm{d}y_a. \end{cases}$$

Denote by Tr the trace of a bilnear form with respect to the flat metric g_a. Then, keeping in mind the formula (3.8), we get

$$\begin{cases} \mathrm{Tr}(\Phi^0) = -\partial_{x_a}\ell_2 - \partial_{y_a}\ell_3 + \mathrm{h}_2^2(\mathrm{k}_2^2 - \mathrm{k}_3^3) + 2\mathrm{h}_3^2\mathrm{k}_3^2, \\ \mathrm{Tr}(\Phi^1) = \partial_{x_a}\mathrm{h}_2^2 + \partial_{y_a}\mathrm{h}_3^2 - \ell_2 = 0, \\ \mathrm{Tr}(\Phi^2) = \partial_{x_a}\mathrm{h}_3^2 - \partial_{y_a}\mathrm{h}_2^2 - \ell_3 = 0, \\ \mathrm{Tr}(\Phi^4) = 0. \end{cases}$$

Hence, the tension field of the conformal Gauss map is given by

$$\tau = -(\partial_{x_a}\ell_2 + \partial_{y_a}\ell_3 - \mathrm{h}_2^2(\mathrm{k}_2^2 - \mathrm{k}_3^3) - 2\mathrm{h}_3^2\mathrm{k}_3^2)\mathrm{A}_0^a = -4\mathrm{Re}(\partial_{\overline{z}_a}\ell_a - \frac{1}{2}\mathrm{h}_a\mathrm{k}_a)\mathrm{A}_0^a.$$

Taking into account the last identity in (3.8), it follows that $\tau = 0$ if and only if $\partial_{\overline{z}_a}\ell_a - \mathrm{h}_a\mathrm{k}_a/2 = 0$. Then, by Theorem 3.12, it follows that f is a space-like Willmore immersion if and only if $\partial_{\overline{z}_a}\ell_a = \mathrm{h}_a\mathrm{k}_a/2$. □

Since

$$\alpha_4^2 + i\alpha_4^3 = \mathrm{k}_a \mathrm{d}\overline{z}_a + \frac{1}{2}|\mathrm{h}_a|^2 \mathrm{d}z_a$$

we can reformulate Lemma 4.1 as follows :

Corollary 4.2. *$f : \mathcal{S} \to \mathcal{E}$ is a space-like Willmore immersion if and only if*

$$d\ell_a = \frac{1}{2}\mathrm{h}_a\mathrm{k}_a d\overline{z}_a + \partial_{z_a}\ell_a dz_a = \frac{1}{2}\mathrm{h}_a(\alpha_4^2 + i\alpha_4^3) + \mathrm{q}_a dz_a, \tag{4.2}$$

where

$$\mathrm{q}_a = \partial_{z_a}\ell_a - \frac{1}{4}|\mathrm{h}_a|^2\mathrm{h}_a. \tag{4.3}$$

Lemma 4.3. *Let $f : \mathcal{S}' \to \mathcal{E}$ be a space-like Willmore immersion. Then, two possibilities may occur : either the Poynting field Y_f is identically zero or else its zero set $\mathcal{Z}$ is discrete . The first possibility occurs if and only if f is totally umbilical. If f is not totally umbilical, then there exists a unique oriented isotropic line sub-bundle $\mathcal{Y}_f$ of $\mathcal{K}^3_f \otimes (\mathcal{S} \times \mathbb{R}^{2,3})$ such that $\mathrm{Y}_f|_p \in \mathcal{Y}_f|_p$, for every $p \in \mathcal{S}$ and, in addition, $\mathrm{Y}_f|_{\mathcal{S}-\mathcal{Z}}$ is a positive-oriented trivialization of $\mathcal{Y}_f$. If $\mathcal{S}$ is compact and f is not totally umbilical, then $\mathcal{Z}$ is finite and the zeroes of Y_f are of finite order.*

Proof. Recall that

$$\mathrm{Y}_f|_{U_a} = (\mathrm{A}^a_0)^3 \otimes (2|\ell_a|^2\mathrm{A}^a_0 - \ell_a\overline{\mathrm{h}}_a(\mathrm{A}^a_2 - i\mathrm{A}^a_3) - \overline{\ell}_a\mathrm{h}_a(\mathrm{A}^a_2 + i\mathrm{A}^a_3) + |\mathrm{h}_a|^2\mathrm{A}^a_4).$$

From the first identity in (3.8) and from (4.2) we have

$$\partial_{\overline{z}_a}\mathrm{h}_a = \ell_a, \quad \partial_{\overline{z}_a}\ell_a = \frac{1}{2}\mathrm{h}_a\mathrm{k}_a. \tag{4.4}$$

Then, $\mathrm{V}_a =^t (\mathrm{h}_a, \ell_a) : U_a \to \mathbb{C}^2$ is a solution of the linear system

$$\partial_{\overline{z}_a}\mathrm{V}_a = \begin{bmatrix} 0 & 1 \\ \frac{\mathrm{k}_a}{2} & 0 \end{bmatrix} \cdot \mathrm{V}_a.$$

By Proposition 1 of the Appendix, two cases may occur :

- $\mathrm{Y}_f|_{U_a} = 0$;
- for every point $p \in U_a$ there exist an open neighborhood $\widetilde{U} \subset U_a$ of p, a non-zero holomorphic function $\phi : \widetilde{U} \to \mathbb{C}$ and a cross section $\widetilde{\mathrm{Y}} : \widetilde{U} \to \mathcal{K}^3_f \otimes (\mathcal{S} \times \mathbb{R}^{2,3})$ such that

$$\mathrm{Y}_f|_{U_a} = \phi\widetilde{\mathrm{Y}}, \quad \widetilde{\mathrm{Y}}|_p \neq 0.$$

Let $m(p) \in \mathbb{N}$ be the order of vanishing of ϕ at p. Notice that $m(p)$ doesn't depend on the holomorphic chart containing p. If p is contained in a holomorphic chart such that $\mathrm{Y}_f|_{U_a} = 0$, we put $m(p) = \infty$. Then, $\mathcal{Z}$ is the disjoint union of two subsets

$$\mathcal{Z}' = \{p \in \mathcal{Z} : 1 \leq m(p) < \infty\}, \quad \mathcal{Z}'' = \{p \in \mathcal{Z} : m(p) = \infty\}.$$

It is easily seen that $\mathcal{Z}''$ is both closed and open. Consequently, either $\mathcal{Z}'' = \mathcal{S}$ or else $\mathcal{Z}'' = \emptyset$. The first case may occur if and only if $\mathcal{N}_f$ is a constant unit time-like vector of $\mathbb{R}^{2,3}$. If this is the case, $f(\mathcal{S})$ is contained in the totally umbilical round 2-sphere

$$\mathbb{S}^{\uparrow}_{\mathcal{N}_f} = \{|[\mathrm{V}]| \in \mathcal{E}/\langle \mathrm{V}, \mathcal{N}_f\rangle = 0, |[\mathrm{V} \wedge \mathcal{N}_f]| \in \mathcal{L}^{\uparrow}\}.$$

If Y_f is non-zero, then $\mathcal{Z}$ is a discrete set. If $\mathcal{S}$ is compact, $\mathcal{Z}$ is finite and it can be viewed as the support set of the divisor $\sum_{p\in\mathcal{Z}} m(p)p$. Let p_* be one of the isolated zeroes of Y_f and (U_a, z_a) be a holomorphic chart such that $U_a \cap \mathcal{Z} = \{p_*\}$. If necessary, by shrinking U_a, we have

$$\ell_a = \phi \widetilde{\ell}_a, \quad \mathrm{h}_a = \phi \widetilde{\mathrm{h}}_a,$$

where $\phi : U_a \to \mathbb{C}$ is a holomorphic function, non-zero at each $p \in U_a$, $p \neq p_*$ and $(\widetilde{\ell}_a, \widetilde{\mathrm{h}}_a)|_p \neq 0$, for every $p \in U_a$. We say that (U_a, z_a) is adapted to $p_* \in \mathcal{Z}$ and we denote by $J(p_*)$ the set of all $a \in J$ such that (U_a, z_a) is a holomorphic chart adapted to p_*. Let p_* be a zero of Y_f and $a \in J(p_*)$. Then we put

$$\widetilde{\mathrm{Y}}_a = (\mathrm{A}^a_0)^3 \otimes (2|\widetilde{\ell}_a|^2 \mathrm{A}^a_0 - \widetilde{\ell}_a \overline{\widetilde{\mathrm{h}}}_a (\mathrm{A}^a_2 - i\mathrm{A}^a_3) - \overline{\widetilde{\ell}}_a \widetilde{\mathrm{h}}_a (\mathrm{A}^a_2 + i\mathrm{A}^a_3) + |\widetilde{\mathrm{h}}_a|^2 \mathrm{A}^a_4).$$

If $a, b \in J(p_*)$, then $\widetilde{\mathrm{Y}}_b = r^a_b \widetilde{\mathrm{Y}}_a$, where r^a_b is a strictly positive real-valued smooth function. Furthermore, $\widetilde{\mathrm{Y}}_a|_{U_a - \{p_*\}}$ is a positive multiple of $\mathrm{Y}_f|_{U_a-\{p_*\}}$, for every $p_* \in \mathcal{Z}$ and every $a \in J(p_*)$. If $p \notin \mathcal{Z}$, we denote by $|[\mathcal{Y}_p]|$ the oriented line of $\mathcal{K}^3_f|_p \otimes \mathbb{R}^{2,3}$ spanned by $\mathrm{Y}_f|_p$ and, if $p \in \mathcal{Z}$, we denote by $|[\mathcal{Y}_p]|$ the ray spanned by $\widetilde{\mathrm{Y}}_a|_p$, $a \in J(p)$. Then,

$$\mathcal{Y} = \{(p, \mathrm{V}) \in \mathcal{S} \times (\mathcal{K}_f|^3_p \otimes \mathbb{R}^{2,3}) : \mathrm{V} \in |[\mathcal{Y}]|_p\}$$

is the oriented isotropic line bundle satisfying the required properties. $\square$

Corollary 4.4. *Let $f : \mathcal{S} \to \mathcal{E}$ be a non totally umbilical space-like Willmore immersion. Then its dual map can be extended smoothly across the zeroes of the Poynting field.*

Proof. Let (U_a, z_a) be a holomorphic chart such that $U_a \cap \mathcal{Z} = \emptyset$. Then $\widehat{f}|_{U_a} = |[\mathrm{Y}_a]|$. If $U_a \cap \mathcal{Z} = \{p\}$ we choose $a \in J(p)$ and we extend $\widehat{f}|_{U_a - \{p\}}$ by

$$\widehat{f}|_{U_a} = |[(A^a_0)^{-3} \otimes \widetilde{\mathrm{Y}}_a]|.$$

$\square$

Lemma 4.5. *If $f : \mathcal{S} \to \mathcal{E}$ is a space-like Willmore immersion, then $\mathcal{Q}_f$ is a holomorphic differential. In particular, if $\mathcal{S}$ is compact, then $\mathcal{Q}_f = 0$.*

P r o o f. Recall that

$$
(4.5)\qquad \begin{cases} \mathrm{dh}_a = \zeta_a \mathrm{d}z_a + \ell_a \mathrm{d}\overline{z}_a, \\ \mathrm{d}\ell_a = \frac{1}{2}\mathrm{h}_a\mathrm{k}_a \mathrm{d}\overline{z}_a + \partial_{z_a}\ell_a \mathrm{d}z_a = \dfrac{1}{2}\mathrm{h}_a(\alpha_4^2 + i\alpha_4^3) + \mathrm{q}_a \mathrm{d}z_a, \end{cases}
$$

where

$$
(4.6)\quad \zeta_a = \partial_{z_a}\mathrm{h}_a, \quad \mathrm{q}_a = \partial_{z_a}\ell_a - \frac{1}{4}|\mathrm{h}_a|^2\mathrm{h}_a, \quad \alpha_4^2 + i\alpha_4^3 = \mathrm{k}_a \mathrm{d}\overline{z}_a + \frac{1}{2}|\mathrm{h}_a|^2 \mathrm{d}z_a.
$$

Step I. We prove that

$$
\langle \partial^2_{z_a z_a}\mathcal{N}_f, \partial^2_{z_a z_a}\mathcal{N}\rangle|_{U_a} = (\mathrm{h}_a\mathrm{q}_a - \ell_a\zeta_a).
$$

Differentiating $\mathcal{N}_f|_{U_a} = \mathrm{A}_1^a$ we have

$$
\mathrm{d}\mathcal{N}_f|_{U_a} = -\alpha_4^1\mathrm{A}_0^a + \alpha_1^2\mathrm{A}_2^a + \alpha_1^3\mathrm{A}_3^a.
$$

Since

$$
(4.7)\qquad \alpha_4^1 = \ell_a \mathrm{d}z_a + \overline{\ell}_a \mathrm{d}\overline{z}_a, \quad \alpha_1^2 - i\alpha_1^3 = \mathrm{h}_a \mathrm{d}z_a
$$

we get

$$
\partial_{z_a}\mathcal{N}_f|_{U_a} = -\ell_a\mathrm{A}_0^a + \frac{1}{2}\mathrm{h}_a(\mathrm{A}_2^a + i\mathrm{A}_3^a).
$$

Recalling that

$$
\begin{cases} \mathrm{dA}_0^a = \mathrm{d}x_a\mathrm{A}_2^a + \mathrm{d}y_a\mathrm{A}_3^a, \\ \mathrm{dA}_2^a = \alpha_4^2\mathrm{A}_0^a + \alpha_1^2\mathrm{A}_1^a + \mathrm{d}x_a\mathrm{A}_4^a, \\ \mathrm{dA}_3^a = \alpha_4^3\mathrm{A}_0^a + \alpha_1^3\mathrm{A}_1^a + \mathrm{d}y_a\mathrm{A}_4^a, \end{cases}
$$

and keeping in mind (4.5) and (4.7) we obtain[5]

$$
\begin{aligned} \mathrm{d}(\partial_{z_a}\mathcal{N}_f)|_{U_a} \equiv \Bigg(\Big(-\partial_{z_a}\ell_a + \frac{1}{4}\,\mathrm{h}_a|\mathrm{h}_a|^2\Big)\,\mathrm{A}_0^a \\ -\frac{\ell_a}{2}\,(\mathrm{A}_2^a - i\mathrm{A}_3^a) + \frac{\zeta_a}{2}\,(\mathrm{A}_2^a + i\mathrm{A}_3^a) + \frac{1}{2}\,\mathrm{h}_a\mathrm{A}_4^a\Bigg)\,\mathrm{d}z_a. \end{aligned}
$$

Hence,

$$
\partial^2_{z_a z_a}\mathcal{N}_f|_{U_a} = -\mathrm{q}_a\mathrm{A}_0^a - \frac{\ell_a}{2}(\mathrm{A}_2^a - i\mathrm{A}_3^a) + \frac{\zeta_a}{2}(\mathrm{A}_2^a + i\mathrm{A}_3^a) + \frac{\mathrm{h}_a}{2}\mathrm{A}_4^a.
$$

[5] $\equiv$ means equality mod $d\overline{z}_a$.

This implies that $\langle \partial^2_{z_a z_a}\mathcal{N}_f, \partial^2_{z_a z_a}\mathcal{N}\rangle|_{U_a} = (\mathrm{h}_a \mathrm{q}_a - \ell_a\zeta_a)$.

Step II. We show that $\mathrm{h}_a\mathrm{q}_a - \ell_a\zeta_a$ is a holomorphic function. To this end we use (4.5) and

$$\mathrm{d}(\alpha^2_4 + i\alpha^3_4) = \alpha^1_4 \wedge (\alpha^2_1 + i\alpha^3_1) = \overline{\mathrm{h}}_a \ell_a \mathrm{d}z_a \wedge \mathrm{d}\overline{z}_a. \tag{4.8}$$

Taking the exterior derivative of the first identity in (4.5) we have $(\mathrm{d}\zeta_a - \partial_{z_a}\ell_a \mathrm{d}\overline{z}_a) \wedge \mathrm{d}z_a = 0$. Hence we can write

$$\mathrm{d}\zeta_a = L_a \mathrm{d}z_a + \partial_{z_a}\ell_a \mathrm{d}\overline{z}_a, \tag{4.9}$$

where $L_a : U_a \to \mathbb{C}$ is a smooth function. Differentiating the second identity in (4.5), using the first identity in (4.5) and (4.8) we find

$$\begin{aligned} 0 &= \frac{1}{2}\mathrm{dh}_a \wedge (\mathrm{k}_a \mathrm{d}\overline{z}_a + \frac{|\mathrm{h}_a|^2}{2}\mathrm{d}z_a) + \frac{|\mathrm{h}_a|^2\ell_a}{2}\mathrm{d}z_a \wedge \mathrm{d}\overline{z}_a + \mathrm{dq}_a \wedge \mathrm{d}z_a \\ &= \frac{1}{2}(\ell_a d\overline{z}_a + \zeta_a \mathrm{d}z_a) \wedge (\mathrm{k}_a \mathrm{d}\overline{z}_a + \frac{|\mathrm{h}_a|^2}{2}\mathrm{d}z_a) + (\mathrm{dq}_a - \frac{|\mathrm{h}_a|^2\ell_a}{2}\mathrm{d}\overline{z}_a) \wedge \mathrm{d}z_a \\ &= (\mathrm{dq}_a - \frac{|\mathrm{h}_a|^2\ell_a}{4}\mathrm{d}\overline{z}_a - \frac{\zeta_a \mathrm{k}_a}{2}\mathrm{d}\overline{z}_a) \wedge \mathrm{d}z_a. \end{aligned}$$

This implies

$$\mathrm{dq}_a = M_a \mathrm{d}z_a + \frac{1}{2}(\zeta_a \mathrm{k}_a + \frac{1}{2}\ell_a|\mathrm{h}_a|^2)\mathrm{d}\overline{z}_a, \tag{4.10}$$

where $M_a : U_a \to \mathbb{C}$ is a smooth function. Combining (4.5), (4.9) and (4.10) we obtain

$$\begin{aligned} \partial_{\overline{z}_a}(\mathrm{q}_a\mathrm{h}_a - \ell_a\zeta_a) &= \frac{\mathrm{h}_a}{2}(\zeta_a \mathrm{k}_a + \frac{1}{2}|\mathrm{h}_a|^2\ell_a) + \mathrm{q}_a\ell_a - \frac{\zeta_a \mathrm{k}_a \mathrm{h}_a}{2} - \ell_a\partial_{z_a}\ell_a \\ &= \ell_a(\mathrm{q}_a + \frac{1}{4}\mathrm{h}_a|\mathrm{h}_a|^2 - \partial_{z_a}\ell_a) = 0. \end{aligned}$$

If $\mathcal{S}$ is compact, then by Remark 3.1, $\mathcal{S}$ is biholomorphically equivalent to the Riemann sphere. But the Riemann-Roch Theorem implies that every abelian differential on $\mathbb{S}^2$ is identically zero. □

Remark 4.6. From the proof of the previous Lemma it follows that if $\mathcal{S}$ is compact and (U_a, z_a) is a holomorphic chart, then

$$\mathrm{q}_a\mathrm{h}_a - \ell_a\zeta_a = 0.$$

Lemma 4.7. *If $\mathcal{S}$ is compact and $f : \mathcal{S} \to \mathcal{E}$ is a non-totally umbilical space-like Willmore immersion then the dual map of f is constant.*

Proof. It suffices to prove that the dual is constant on $\mathcal{S} - \mathcal{Z}$. If (U_a, z_a) is a holomorphic chart such that $U_a \cap \mathcal{Z} = \emptyset$ then $\widehat{f}|_{U_a} = |[\mathrm{Y}_a]|$ where Y_a is as in (3.13). Then, we need to show that $\mathrm{Y}_a \wedge \mathrm{dY}_a = 0$, for every holomorphic chart such that $U_a \cap \mathcal{Z} = \emptyset$. Using (4.5), (4.6) and taking into account the following formulas

$$\begin{aligned}
\mathrm{dA}_0^a &= \frac{1}{2}\mathrm{d}z_a(\mathrm{A}_2^a - i\mathrm{A}_3^a) + \frac{1}{2}\mathrm{d}\overline{z}_a(\mathrm{A}_2^a + i\mathrm{A}_3^a)\\
\mathrm{d}(\mathrm{A}_2^a - i\mathrm{A}_2^a) &= (\alpha_4^2 - i\alpha_4^3)\mathrm{A}_0^a + (\alpha_1^2 - i\alpha_1^3)\mathrm{A}_1^a + \mathrm{d}\overline{z}_a\mathrm{A}_4^a\\
&= (\overline{\mathrm{k}}_a\mathrm{d}z_a + \frac{1}{2}|\mathrm{h}_a|^2\mathrm{d}\overline{z}_a)\mathrm{A}_0^a + \mathrm{h}_a\mathrm{d}z_a\mathrm{A}_1^a + \mathrm{d}\overline{z}_a\mathrm{A}_4^a,\\
\mathrm{dA}_4^a &= \alpha_4^1\mathrm{A}_1^a + \alpha_4^2\mathrm{A}_2^a + \alpha_4^3\mathrm{A}_3^a = (\ell_a\mathrm{d}z_a + \overline{\ell}_a\mathrm{d}\overline{z}_a)\mathrm{A}_1^a\\
&\quad + (\frac{\overline{\mathrm{k}}_a}{2}\mathrm{d}z_a + \frac{|\mathrm{h}_a|^2}{4}\mathrm{d}\overline{z}_a)(\mathrm{A}_2^a + i\mathrm{A}_3^a) + (\frac{\mathrm{k}_a}{2}\mathrm{d}\overline{z}_a + \frac{|\mathrm{h}_a|^2}{4}\mathrm{d}z_a)(\mathrm{A}_2^a - i\mathrm{A}_3^a)
\end{aligned}$$

we obtain

$$\mathrm{dY}_a = \mathrm{S}_a\mathrm{d}z_a + \overline{\mathrm{S}}_a\mathrm{d}\overline{z}_a,$$

where

$$\mathrm{S}_a = 2\overline{\ell}_a\mathrm{q}_a\mathrm{A}_0^a - \overline{\mathrm{h}}_a\mathrm{q}_a(\mathrm{A}_2^a - i\mathrm{A}_3^a) - \zeta_a\overline{\ell}_a(\mathrm{A}_2^a + i\mathrm{A}_3^a) + \zeta_a\overline{\mathrm{h}}_a\mathrm{A}_4^a. \tag{4.11}$$

We then have

$$\begin{aligned}
\mathrm{Y}_a \wedge \mathrm{S}_a &= (2|\ell_a|^2\mathrm{A}_0^a - \ell_a\overline{\mathrm{h}}_a(\mathrm{A}_2^a - i\mathrm{A}_3^a) - \overline{\ell}_a\mathrm{h}_a(\mathrm{A}_2^a + i\mathrm{A}_3^a) + |\mathrm{h}_a|^2\mathrm{A}_4^a)\wedge\\
&\quad (2\overline{\ell}_a\mathrm{q}_a\mathrm{A}_0^a - \overline{\mathrm{h}}_a\mathrm{q}_a(\mathrm{A}_2^a - i\mathrm{A}_3^a) - \zeta_a\overline{\ell}_a(\mathrm{A}_2^a + i\mathrm{A}_3^a) + \zeta_a\overline{\mathrm{h}}_a\mathrm{A}_4^a)\\
&= (\mathrm{h}_a\mathrm{q}_a - \ell_a\zeta_a)(-2\overline{\ell}_a\overline{\mathrm{h}}_a\mathrm{A}_0^a \wedge \mathrm{A}_4^a + 2\overline{\ell}_a^2\mathrm{A}_0^a \wedge (\mathrm{A}_2^a + i\mathrm{A}_3^a)\\
&\quad + \overline{\ell}_a\overline{\mathrm{h}}_a(\mathrm{A}_2^a + i\mathrm{A}_3^a) \wedge (\mathrm{A}_2^a - i\mathrm{A}_3^a) - \overline{\mathrm{h}}_a^2\mathrm{A}_4^a \wedge (\mathrm{A}_2^a - i\mathrm{A}_3^a)) = 0.
\end{aligned}$$

We have thus proved the result. □

End of the proof. Suppose, by contradiction, that $f : \mathcal{S} \to \mathcal{E}$ is a non-totally umbilical Willmore immersion of a compact surface in $\mathcal{E}$. Since $\widehat{f}$ is constant, then $\widehat{f} = |[\mathrm{V}]|$, where V is a non-zero isotropic vector. If $\mathbf{B} \in \mathrm{M}_+^{\uparrow}$, then $\mathbf{B} \cdot \widehat{f}$ is the dual map of $\mathbf{B} \cdot f$. Hence, possibly replacing f with $\mathbf{B} \cdot f$, for some $\mathbf{B} \in \mathrm{M}_+^{\uparrow}$, we may assume that V is the last column vector E_4 of the standard basis of $\mathbb{R}^{2,3}$. Let Δ be the finite set $f^{-1}(|[\mathrm{E}_4]|)$ and $\mathcal{S}'$ be the complement of Δ. Put $J' = \{a \in J : U_a \cap \Delta = \emptyset\}$. For every $a \in J'$ the first column vector A_0^a is a lift of f. Hence $|[\mathrm{A}_0^a]| \neq |[\mathrm{E}_4^a]|$. Since $|[\mathrm{E}_4^a]| = |[\mathrm{Y}_a]|$, then $|[\mathrm{A}_0^a|_p]| \neq |[\mathrm{Y}_a|_p]|$, for every $p \in U_a$. This implies that $|\mathrm{h}_a|^2 > 0$, i.e. the points of $\mathcal{S}'$ are non-umbilical. For every $a \in J'$ there exists a strictly positive function ρ_a such that $\mathrm{E}_4^a = \rho_a \mathrm{Y}_a$. We write

$$\rho_a \mathrm{Y}_a = \mathrm{r}_0^a \mathrm{A}_0^a + \mathrm{x}_2^a \mathrm{A}_2^a + \mathrm{y}_3^a \mathrm{A}_3^a + \mathrm{r}_4^a \mathrm{A}_4^a,$$

where $\mathrm{r}_0^a, \mathrm{x}_2^a, \mathrm{x}_3^a$ and $\mathrm{r}_4^a = \rho_a |\mathrm{h}_a|^2 > 0$ are smooth functions. We consider the smooth map $\mathbf{B}_a : U_a \to \mathrm{M}_+^{\uparrow}$ defined by $\mathbf{B}_a = \mathbf{B}(1/r_4^a, 0, \mathrm{x}_2^a/r_4^a, \mathrm{y}_3^a/r_4^a)$. Then we put $\widehat{\mathbf{A}}_a = \mathbf{A}_a \cdot \mathbf{B}_a$. This is a second-order frame along f such that $\widehat{\mathrm{A}}_4^a = \mathrm{E}_4^a$. Therefore, the $\mathfrak{m}$-valued 1-form $\widehat{\alpha} = \widehat{\mathbf{A}}_a^{-1} \mathrm{d}\widehat{\mathbf{A}}_a$ is as in (3.3) with

$$\widehat{\alpha}_0^0 = \widehat{\alpha}_4^1 = \widehat{\alpha}_4^2 = \widehat{\alpha}_4^3 = 0. \tag{4.12}$$

In addition, $\xi_a = \widehat{\alpha}_0^2 + i\widehat{\alpha}_0^3$ is of type $(1,0)$, non-zero at every point $p \in U_a$ and $\widehat{\alpha}_1^2 - i\widehat{\alpha}_1^3 = \widehat{\mathrm{h}}_a \xi_a$, where $\widehat{\mathrm{h}}_a = \widehat{\mathrm{h}}_{a,2}^2 - i\widehat{\mathrm{h}}_{a,3}^2 : U_a \to \mathbb{C}$ is a complex-valued smooth function. For each $p \in \mathcal{S}'$ we put $J'(p) = \{a \in J' : p \in U_a\}$. Let $\widehat{\mathcal{F}}|_p$ be the set of all $\mathbf{X} \in \mathrm{M}_+^{\uparrow}$ such that $\mathbf{X} = \widehat{\mathbf{A}}_a|_p$, for some $a \in J'(p)$. Then

$$\widehat{\mathcal{F}} = \{(p, \mathbf{X}) \in \mathcal{S}' \times \mathrm{M}_+^{\uparrow} : \mathbf{X} \in \widehat{\mathcal{F}}|_p\}$$

is a circle-bundle over $\mathcal{S}'$ and its cross-sections are the second-order frames $\widehat{\mathbf{A}}_a$, $a \in J'$. Let $\widehat{\mathcal{T}} : \widehat{\mathcal{F}} \to \mathrm{M}_+^{\uparrow}$ be the tautological map defined by $\widehat{\mathcal{T}}(p, \mathbf{X}) = \mathbf{X}$ and $\widehat{\tau}$ be the pull-back $\widehat{\mathcal{T}}^*(\mu)$ of the Maurer-Cartan form of $\mathrm{M}_+^{\uparrow}$. From (4.12) it follows that $\widehat{\tau}$ is a 1-form with value in the Lie subalgebra $\widehat{\mathfrak{p}} \subset \mathfrak{m}$. Since $\widehat{\mathcal{F}}$ is connected then, possibly acting on the left of f by an element of $\mathrm{M}_+^{\uparrow}$, we may suppose that the image of $\widehat{\mathcal{T}}$ is contained in the closed subgroup $\widehat{\mathrm{P}}_+^{\uparrow}$. The map

$$\mathcal{P} = \mathbf{J}^{-1} \circ \widehat{\mathcal{T}} : \widehat{\mathcal{F}} \to \mathrm{P}_+^{\uparrow}$$

is constant along the fibers of $\widehat{\mathcal{F}} \to \mathcal{S}'$ and hence it induces a space-like immersion $\mathtt{f} : \mathcal{S}' \to \mathbb{R}^{1,2}$ such that $\mathtt{j} \circ \mathtt{f} = f|_{\mathcal{S}'}$. Notice that $\mathbf{P}_a := \mathcal{P} \circ \widehat{\mathbf{A}}_a : U_a \to \mathrm{P}_+^{\uparrow}$ is a Lorentzian first-order frame field along $\mathtt{f}$ such that $\mathbf{J}^*(\mathbf{P}_a^{-1}\mathrm{d}\mathbf{P}_a) = \widehat{\mathbf{A}}_a^{-1}\mathrm{d}\widehat{\mathbf{A}}_a$. This implies

$$\mathbf{P}_a^{-1}\mathrm{d}\mathbf{P}_a = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\alpha}_1^2 & \widehat{\alpha}_1^3 \\ \widehat{\alpha}_0^2 & \widehat{\alpha}_1^2 & 0 & -\widehat{\alpha}_2^3 \\ \widehat{\alpha}_0^3 & \widehat{\alpha}_1^3 & \widehat{\alpha}_2^3 & 0 \end{bmatrix}.$$

Let $\mathtt{n_f}$ be the future-directed unit time-like normal vector field along $\mathtt{f}$. Using $\widehat{\alpha}_1^2 - i\widehat{\alpha}_1^3 = \widehat{\mathrm{h}}_a(\widehat{\alpha}_0^2 + i\widehat{\alpha}_0^3)$, we have

$$\begin{cases} (\mathrm{d}\mathtt{f}, \mathrm{d}\mathtt{f})|_{U_a} = (\widehat{\alpha}_0^2)^2 + (\widehat{\alpha}_0^3)^2, \\ (\mathrm{d}\mathtt{n_f}, \mathrm{d}\mathtt{f})|_{U_a} = \widehat{\mathrm{h}}_{a,2}^2(\widehat{\alpha}_0^2)^2 + 2\widehat{\mathrm{h}}_{a,2}^2\widehat{\alpha}_0^2\widehat{\alpha}_0^3 - \widehat{\mathrm{h}}_{a,2}^2(\widehat{\alpha}_0^3)^2. \end{cases}$$

Hence, $\mathtt{f}$ is a maximal space-like immersion in the Minkowski space. This implies that $\mathtt{n_f}$ is a holomorphic map from $\mathcal{S}'$ into the hyperbolic plane, identified with the quadric of all future-directed unit time-like vectors of $\mathbb{R}^{1,2}$. On the other hand, the Lorentzian Gauss map $\mathtt{n_f}$ and the conformal Gauss map $\mathcal{N}_f$ of $f = \mathbf{j} \circ \mathtt{f}$ are related by (see Remark 3.17)

$$\mathcal{N}_f =^t (0, \mathtt{n_f}, \mathtt{f}^* \cdot \mathtt{n_f}).$$

Thus, $\mathtt{n_f}$ can be smoothly extended to all of $\mathcal{S}$. Consequently, $\mathtt{n_f}$ gives rise to a holomorphic map from $\mathcal{S}$ to the hyperbolic plane. Since $\mathcal{S}$ is compact, by the Liouville theorem, $\mathtt{n_f}$ is constant. Hence, $\mathtt{f}(\mathcal{S}')$ lies in a space-like plane and and $f(\mathcal{S})$ would be a totally umbilical round 2-sphere of $\mathcal{E}$. We have thus reached a contradiction. $\square$

Let's think of $\mathcal{E}$ as the codimension two submanifold $\mathbb{S}^1 \times \mathbb{S}^2$ of $\mathbb{R}^{2,3}$ (cfr. Definition 2.3). The map

$$\pi : (t, \mathrm{x}) \in \mathcal{E}^* = \mathbb{R}\times\mathbb{S}^2 \to \frac{\cos(t)}{\sqrt{2}}(\mathrm{E}_0+\mathrm{E}_4)+\sin(t)\mathrm{E}_1+\mathrm{x}_1\mathrm{E}_2+\mathrm{x}_2\mathrm{E}_3+\frac{\mathrm{x}_3}{\sqrt{2}}(\mathrm{E}_4-\mathrm{E}_0),$$

exhibits the physical Einstein universe $\mathcal{E}^* = \mathbb{R}\times\mathbb{S}^2$, equipped with the conformal structure determined by the Lorentzian pseudo-metric $\ell_{\mathcal{E}^*} = -\mathrm{d}t^2+\mathrm{dx}_1^2+\mathrm{dx}_2^2+\mathrm{dx}_3^2$, as the Lorentzian universal covering space of $\mathcal{E}$. The action of $\mathrm{M}_+^\uparrow$ lifts to an almost effective action of the universal covering group $p_{\mathrm{M}_+^\uparrow} : \widehat{\mathrm{M}}_+^\uparrow \to \mathrm{M}_+^\uparrow$ of $\mathrm{M}_+^\uparrow$ on the left of $\mathcal{E}^*$ such that

$$\pi\left(\widetilde{\mathbf{B}} \cdot (t, \mathrm{x})\right) = p_{\mathrm{M}_+^\uparrow}(\widetilde{\mathbf{B}}) \cdot \pi(t, \mathrm{x}),$$

for every $\widetilde{\mathbf{B}} \in \widehat{\mathrm{M}}_+^\uparrow$ and every $(t, \mathrm{x}) \in \mathcal{E}^*$ (see [**5**]). Moreover, since $\mathrm{M}_+^\uparrow$ acts on $\mathcal{E}$ by conformal transformations, also the action of $\widehat{\mathrm{M}}_+^\uparrow$ preserves the conformal structure of $\mathcal{E}^*$. Notice that the 2-sphere $\widehat{\mathbb{S}}_r^2 = \{(t, \mathrm{x}) \in \mathcal{E}^* : t = r\}$ is totally umbilical and, in addition, every compact, connected space-like totally umbilical surface of $\mathcal{E}^*$ is given by $\widehat{\mathbf{B}} \cdot \widehat{\mathbb{S}}_r^2$, for some $\widehat{\mathbf{B}} \in \widehat{\mathrm{M}}_+^\uparrow$.

As a straighforward consequence of Theorem A, we have

Theorem B. *Let $\mathcal{S}$ be a compact 2-dimensional connected manifold and $\widetilde{f}:\mathcal{S}\to\mathcal{E}^*$ be a space-like Willmore immersion. Then $\widetilde{f}(\mathcal{S})$ is a totally umbilical round 2-sphere of $\mathcal{E}^*$.*

Proof. The composite map $f=\pi\circ\widetilde{f}:\mathcal{S}\to\mathcal{E}$ is a space-like Willmore immersion. Then, by Theorem A, there exists $\mathbf{B}\in \mathrm{M}_+^{\uparrow}$ such that $\mathbf{B}\cdot f(\mathcal{S})=\{\pi(0,\mathrm{x}):\mathrm{x}\in\mathbb{S}^2\}\cong\mathbb{S}^2$. Choose $\widetilde{\mathbf{B}}\in p^{-1}_{\mathrm{M}_+^{\uparrow}}(\mathbf{B})$. Since $\pi(\widetilde{\mathbf{B}}\cdot\widetilde{f})=\mathbf{B}\cdot f$, then

$$\widetilde{\mathbf{B}}\cdot\widetilde{f}(\mathcal{S})\subset\pi^{-1}(\mathbb{S}^2)=\bigcup_{r\in 2\pi\mathbb{Z}}\widehat{\mathbb{S}}^2_r\,.$$

Hence, there exist $r\in 2\pi\mathbb{Z}$ such that $\widetilde{f}(\mathcal{S})=\widetilde{\mathbf{B}}^{-1}\cdot\widehat{\mathbb{S}}^2_r$. This concludes the proof. □

5 - Appendix

Proposition 1. *Let (U,z) be a holomorphic chart on a Riemann surface $\mathcal{S}$, $\mathrm{Q}:U\to\mathbb{C}(n,n)$ be a smooth map and $\mathrm{V}:U\to\mathbb{C}^n$ be a solution of the linear system*

$$\partial_{\overline{z}}\mathrm{V}=\mathrm{Q}\cdot\mathrm{V}.$$

Then, for each point $p\in U$ there exist an open neighborhood $\widehat{U}\subset U$ containing p, a smooth map $\mathrm{M}:\widehat{U}\to\mathrm{GL}(n,\mathbb{C})$ and a holomorphic map $\mathrm{W}:\widehat{U}\to\mathbb{C}^n$ such that $\mathrm{V}|_{\widehat{U}}=\mathrm{M}\cdot\mathrm{W}$. In particular, if V is not identically zero, then the zero locus of V is discrete. Furthermore, if p is an isolated zero of V then, locally, we can write $\mathrm{V}=\psi\cdot\widetilde{\mathrm{W}}$, where ψ is a holomorphic function with a zero of order $m(p)\in\mathbb{N}$ at p and $\widetilde{\mathrm{W}}|_p\neq 0$.

Proof. First we recall a general fact on complex vector bundles [**19**] :

Fact. Let M be a complex manifold and $E\to M$ be a complex vector bundle over M. Denote by $\Lambda(E)$ the space of the complex, E-valued exterior differential forms and by $\Lambda^{(p,q)}(E)$ the space of the E-valued forms of type (p,q). Consider a linear connection $D:\Lambda(E)\to\Lambda(E)$. Using the complex structure we can write $D=D'+D''$, where $D':\Lambda^{(p,q)}(E)\to\Lambda^{(p+1,q)}(E)$ and $D'':\Lambda^{(p,q)}(E)\to\Lambda^{(p,q+1)}(E)$. If [6] $D''\circ D''=0$ then, using the Newlander-Nirenberg Theorem [**30**], one can prove that E possesses a unique structure of a holomorphic vector bundle such that a local cross section $\mathfrak{s}:U\to E$ is holomorphic if and only if $D''\mathfrak{s}=0$.

[6] If M is a Riemann surface this condition is automatically fulfilled.

Let V and Q be as in the statement. Consider the trivial vector bundle $E = U \times \mathbb{C}^n$ with its structure of a complex vector bundle but without any holomorphic structure. For each $j = 1, ..., n$, denote by $\mathfrak{s}_j$ the constant section of E with 1 in its j-th entry and zero elsewhere. On E we consider the covariant derivative

$$D(u^j \mathfrak{s}_j) \to (\mathrm{d}u^j - \mathrm{Q}^j_k u^k)\mathfrak{s}_j \, .$$

On E we put the unique holomorphic structure such that $\mathfrak{s}$ is a holomorphic section if and only if $D''\mathfrak{s} = 0$. Consequently, if V is a solution of the linear system, the cross section $\mathfrak{v} = \mathrm{V}^j \mathfrak{s}_j$ is holomorphic. Given a point $p \in U$ we choose a holomorphic trivialization $(\widetilde{\mathfrak{s}}_1, ..., \widetilde{\mathfrak{s}}_n)$ defined on an open neighborhood $\widehat{U} \subset U$ of p. Then, $\mathfrak{v}|_{\widehat{U}} = \mathrm{w}^j \widetilde{\mathfrak{s}}_j$, where $\mathrm{w}^1, ..., \mathrm{w}^n$ are holomorphic functions. Denote by $\mathrm{M} : \widehat{U} \to \mathrm{GL}(n, \mathbb{C})$ the unique map such that $\widetilde{\mathfrak{s}}_j = \mathrm{M}^k_j \mathfrak{s}_k$. Then

$$\mathfrak{v}|_{\widehat{U}} = v^k \mathfrak{s}_k = \mathrm{w}^j \widetilde{\mathfrak{s}}_j = \mathrm{w}^j \mathrm{M}^k_j \mathfrak{s}_k \, .$$

This implies that $\mathrm{V} =^t (\mathrm{v}^1, ..., \mathrm{v}^n) = \mathrm{M} \cdot^t (\mathrm{w}^1, ..., \mathrm{w}^n) = \mathrm{M} \cdot \mathrm{W}$. Assume that V is not identically zero. Suppose $\mathrm{V}|_p = 0$, then W is a holomorphic function defined near p, not identically zero and vanishing at p. Thus, we can write $\mathrm{W} = \psi \widehat{\mathrm{W}}$, where ψ is a holomorphic function with a zero of order $m(p)$ at p, $\widehat{\mathrm{W}}$ is holomorphic and $\widehat{\mathrm{W}}|_p \neq 0$. Then, $\widetilde{\mathrm{W}} = \mathrm{M} \cdot \widehat{\mathrm{W}}$ is a smooth map such that $\widetilde{\mathrm{W}}|_p \neq 0$ and that $\mathrm{V} = \psi \widetilde{\mathrm{W}}$ near p. □

References

[1] L. J. Alías and B. Palmer, *Conformal geometry of surfaces in Lorentzian space forms*, Geom. Dedicata **60** (1996), no. 3, 301–315.

[2] L. J. Alías and B. Palmer, *Deformations of stationary surfaces*, Classical Quantum Gravity **14** (1997), no. 8, 2107–2111.

[3] T. Barbot, V. Charette, T. Drumm, W. M. Goldman and K. Melnick, *A primer on the* $(2+1)$*-Einstein universe*, in "Recent developments in pseudo-Riemannian geometry", ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008, 179–229.

[4] W. Blaschke, *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie*, B. 3, bearbeitet von G. Thomsen, J. Springer, Berlin, 1929.

[5] G. E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics, **46**, Academic Press, New York-London, 1972.

[**6**] R. L. Bryant, *A duality theorem for Willmore surfaces*, J. Differential Geom. **20** (1984), 23–53.

[**7**] R. L. Bryant, *Surfaces in conformal geometry*, Proc. Sympos. Pure Math., **48**, Amer. Math. Soc., Providence, RI, 1988, 227–240.

[**8**] Y. Deng and C. Wang, *Time-like Willmore surfaces in Lorentzian* 3*-space*, Sci. China Ser. A **49** (2006), no. 1, 75–85.

[**9**] B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, *Modern geometry–methods and applications. Part I*, 2nd ed., Graduate Texts in Mathematics, **93**, Springer-Verlag, New York, 1992.

[**10**] A. Dzhalilov, E. Musso and L. Nicolodi, *Conformal geometry of timelike curves in the* $(2+1)$*-Einstein universe*, Nonlinear Anal. **143** (2016), 224–255.

[**11**] A. Einstein, *Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften, Berlin, 1917, 142–152.

[**12**] J. H. Eshenburg, *Willmore Surfaces and Möbius geometry*, unpublished manuscript, 1984.

[**13**] O. Eshkobilov, E. Musso and L. Nicolodi, *Lorentz manifolds whose restricted conformal group has maximal dimension*, preprint, 2018.

[**14**] C. Frances, *Géometrie et dynamique lorentziennes conformes*, Thése, E.N.S. Lyon, 2002.

[**15**] C. Frances, *Sur les variétés lorentziennes dont le group conforme est essentiel*, Math. Ann. **332** (2005), no. 1, 103–119.

[**16**] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge Monographs on Mathematical Physics, **1**, Cambridge University Press, London–New York, 1973.

[**17**] G. R. Jensen, E. Musso and L. Nicolodi, *Surfaces in classical geometries*, A treatment by moving frames, Universitext, Springer, Cham, 2016.

[**18**] G. R. Jensen, M. Rigoli and K. Yang, *Holomorphic curves in the complex quadric*, Bull. Austral. Math. Soc. **35** (1987), no. 1, 125–148.

[**19**] S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, **15**, Princeton University Press, Princeton, NJ, 1987.

[**20**] F. C. Marques and A. Neves, *Min-max theory and the Willmore conjecture*, Ann. of Math. **179** (2014), no. 2, 683–782.

[**21**] C. Nie, X. Ma and C. Wang, *Conformal CMC-surfaces in Lorentzian space forms*, Chin. Ann. Math. Ser. B **28** (2007), no. 3, 299–310.

[**22**] B. Palmer, *Spacelike constant mean curvature surfaces in pseudo-Riemannian space forms*, Ann. Global Anal. Geom. **8** (1990), no. 3, 217–226.

[**23**] R. Penrose, *Cycles of time. An extraordinary new view of the universe*, Alfred A. Knopf, New York, 2010.

[**24**] R. Penrose, *On the gravitization of quantum mechanics 2: Conformal cyclic cosmology*, Found. Phys. **44** (2014), no. 8, 873-890.

[**25**] J. Rawnsley, *On the universal covering group of the real symplectic group*, J. Geom. Phys. **62** (2012), no. 10, 2044–2058.

[**26**] G. Thomsen, *Über konforme Geometrie I: Grundlagen der konformen flächentheorie*, Abh. Math. Sem. Hamburg **3** (1924), 31–56.

[**27**] P. Tod, *Penrose's Weyl curvature hypothesis and conformally-cyclic cosmology*, J. Phys.: Conf. Ser. **229** (2010), no. 1, 012013

[**28**] P. Wang, *Blaschke's problem for timelike surfaces in pseudo-Riemannian space forms*, Int. J. Geom. Methods Mod. Phys. **7** (2010), no. 7, 1147–1158.

[**29**] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, **94**, Springer-Verlag, New York-Berlin, 1983.

[**30**] R. O. Wells, Jr., *Differential analysis on complex manifolds*, 2nd ed., Graduate Texts in Mathematics, **65**, Springer-Verlag, New York-Berlin, 1980.

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