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On m-S-complemented subgroups of finite groups

Abstract. Let G be a finite group and H a subgroup of G. We say that H: is *generalized S-quasinormal in G* if $H = \langle A, B\rangle$ for some modular subgroup A and S-quasinormal subgroup B of G; *m-S-complemented* in G if there are a generalized S-quasinormal subgroup S and a subgroup T of G such that $G = HT$ and $H \cap T \leq S \leq H$. In this paper, we study finite groups with given systems of m-S-complemented subgroups. In particular, we prove the following result: Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups, and let E be a normal subgroup of a finite group G such that $G/E \in \mathfrak{F}$. If for any Sylow subgroup P of E every maximal subgroup of P not having a supersoluble supplement in G is m-S-complemented in G, then $G \in \mathfrak{F}$.

Keywords. Finite group, modular subgroup, S-quasinormal subgroup, generalized S-quasinormal subgroup, m-S-complemented subgroup.

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1 - Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\pi(G)$ is the set of all primes dividing the order $|G|$ of G; C_n denotes a cyclic group of order n.

A subgroup M of G is called *modular* if M is a modular element (in the sense of Kurosh [**23**, p. 43]) of the lattice $\mathcal{L}(G)$ of all subgroups of G, that is,

(i) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \leq G, Z \leq G$ such that $X \leq Z$, and

(ii) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \leq G, Z \leq G$ such that $M \leq Z$.

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A subgroup H of G is said to be *S-quasinormal* [**17**] or *S-permutable* [**5**,**10**] in G if H permutes with every Sylow subgroup P of G, that is, $HP = PH$. The subgroup H of G is said to be *generalized S-quasinormal* in G [**13**] if there are a modular subgroup A and an S-quasinormal subgroup B of G such that $H = \langle A, B\rangle$.

Interesting applications of generalized S-quasinormal subgroups were discussed in the paper [**13**]. In this paper, we consider the following generalization of such subgroups.

Definition 1.1. We say that a subgroup H of G is *m-S-complemented in G* if there are a generalized S-quasinormal subgroup S and a subgroup T of G such that $G = HT$ and $H \cap T \leq S \leq H$.

It is clear that every generalized S-quasinormal subgroup is m-S-complemented. Every modular subgroup and every S-quasinormal subgroup are generalized S-quasinormal. Now consider the following

Example 1.2. Let $C_3 \wr A_4 = P \rtimes A_4$, where A_4 is the alternating group of degree 4 and P is the base group of the regular wreath product $C_3 \wr A_4$. Let $G = (P \rtimes A_4) \times (C_{11} \rtimes C_5)$, where $C_{11} \rtimes C_5$ is a non-abelian group of order 55. Let Q be the Sylow 2-subgroup of A_4 and R a Sylow 3-subgroup of A_4. Then PQ is supersoluble, so some subgroup B of P with $|B| = 3$ is normal in PQ. Then for every Sylow 3-subgroup G_3 of G we have $B \leq P \leq G_3$, so $BG_3 = G_3 = G_3B$. On the other hand, for every Sylow 2-subgroup Q^x of G we have $Q^x \leq PQ$, so $BQ^x = Q^xB$. Hence B is S-quasinormal in G. In view of [**23**, Theorem 5.1.9], $A = C_5$ is modular in G. Then $S = \langle A, B\rangle = AB$ is generalized S-quasinormal in G.

Now let $H = (AB)Q = A \times BQ$ and $T = PRC_{11}$. Then $G = HT$ and $H \cap T = (AB)Q \cap PRC_{11} = B(AQ \cap PRC_{11}) = B \leq H$. Hence H is m-S-complemented in G.

Next we show that H is not generalized S-quasinormal in G. First note that $H_G = 1$, so for every modular subgroup V of H we have $V^G \leq C_{11} \rtimes C_5$ by Lemma 2.4 below. Therefore A is the largest modular subgroup of H. Assume that H is generalized S-quasinormal in G and let W be an S-quasinormal subgroup of G such that $H = \langle A, W\rangle = AW$. Then $W_G = 1$, so W is a nilpotent subnormal subgroup of G by [**5**, Theorem 1.2.17]. Hence for a Sylow 2-subgroup Q_1 of W we have $1 < Q_1 \leq O_2(G) \leq P \rtimes (Q \rtimes C_p)$ and so $Q_1 \leq C_G(P)$, a contradiction. Therefore H is not generalized S-quasinormal in G.

A subgroup H of G is said to be *complemented* (respectively *c-supplemented* [**8**]) in G, if there is a subgroup T of G such that $G = HT$ and $H \cap T = 1$ (respectively $G = HT$ and $H \cap T \leq H_G$). It is clear that every complemented subgroup and every c-supplemented subgroup are m-S-complemented.

A subgroup H of G is said to be *S-supplemented* [**25**] (respectively *m-supplemented* [**30**]) in G, if there are an S-quasinormal subgroup (respectively a modular subgroup) S and a subgroup T of G such that $G = HT$ and $H \cap T \leq S \leq H$. Every S-supplemented subgroup and every m-supplemented subgroup are m-S-complemented.

Let $K \leq H$ be normal subgroups of G. Then we say, following [**23**], that H/K is *hypercyclically embedded in* G if every chief factor of G between H and K is cyclic. We say also that H is hypercyclically embedded in G if $H/1$ is hypercyclically embedded in G.

Hypercyclically embedded subgroups play an important role in the theory of soluble groups (see the books [**5, 10, 23**]) and the conditions under which a normal subgroup is hypercyclically embedded were found by many authors (see, for example, the recent papers [**24**]– [**14**]).

In this paper, we prove the following results in this line research.

Theorem 1.3. *Let E be a normal subgroup of G and let P be a Sylow p-subgroup of E, where p is the smallest prime dividing $|E|$. If every maximal subgroup of P not having a supersoluble supplement in G is m-S-complemented in G, then E is p-nilpotent and $E/O_{p'}(E)$ is hypercyclically embedded in G.*

Theorem 1.4. *Let E be a normal subgroup of G. Suppose that for any Sylow subgroup P of E every maximal subgroup of P not having a supersoluble supplement in G is m-S-complemented in G, Then E is hypercyclically embedded in G.*

As a first application of Theorem 1.4, we prove also the following result, which covers many known results (see Section 4 below).

Theorem 1.5. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups, and let $X \leq E$ be normal subgroups of G with $G/E \in \mathfrak{F}$. Suppose that for any Sylow subgroup P of X every maximal subgroup of P not having a supersoluble supplement in G is m-S-complemented in G. If $X = E$ of $X = F^*(E)$, then $G \in \mathfrak{F}$.*

In this theorem $X = F^*(E)$ denotes the generalized Fitting subgroup of E [**16**, Ch. X], that is, the product of all normal quasinilpotent subgroups of E.

2 - Preliminaries

The first lemma collects the properties of S-quasinormal subgroups used in our proofs.

Lemma 2.1 (See Chapter 1 in [**5**]). *Let A, B and N be subgroups of G, where A is S-quasinormal in G and N is normal in G.*

(1) *AN/N is S-quasinormal in G/N.*

(2) *If $A \leq B$, then A is S-quasinormal in B.*

(3) *If $N \leq B$ and B/N is S-quasinormal in G/N, then B is S-quasinormal in G.*

(4) *A is subnormal in G and A^G/A_G is nilpotent.*

(5) *If B is S-quasinormal in G, then $A \cap B$ and $\langle A, B\rangle$ are S-quasinormal in G.*

Lemma 2.2. *Let A, B and N be subgroups of G, where A is generalized S-quasinormal in G and N is normal in G. Then:*

(1) *AN/N is generalized S-quasinormal in G/N.*

(2) *If $A \leq B$, then A is generalized S-quasinormal in B.*

(3) *If $N \leq B$ and B/N is generalized S-quasinormal in G/N, then B is generalized S-quasinormal in G.*

(4) *If B is generalized S-quasinormal in G, then $\langle A, B\rangle$ is generalized S-quasinormal in G.*

Proof. Let $A = \langle L, T\rangle$, where L is modular and T is S-quasinormal subgroups of G.

(1) $AN/N = \langle LN/N, TN/N\rangle$, where LN/N is modular in G/N by Property (3) in [**23**, p. 201] and TN/N is S-quasinormal in G/N by Lemma 2.1(1). Hence AN/N is generalized S-quasinormal in G/N.

(2) This follows from Property (2) in [**23**, p. 201] and Lemma 2.1(2) in [**5**].

(3) Let $B/N = \langle V/N, W/N\rangle$, where V/N is modular in G/N and W/N is S-quasinormal in G/N. Then $B = \langle V, W\rangle$, where V is modular in G by Property (4) in [**23**, p. 201] and W is S-quasinormal in G by Lemma 2.1(3). Hence B is generalized S-quasinormal in G.

(4) This follows from Property (5) in [**23**, p. 201] and Lemma 2.1(5). □

Lemma 2.3. *Let A, B and N be subgroups of G, where A is m-S-complemented in G and N is normal in G.*

(1) *If either $N \leq A$ or $(|A|, |N|) = 1$, then AN/N is m-S-complemented in G/N.*

(2) *If $A \leq B$, then A is m-S-complemented in B.*

(3) *If $N \leq B$ and B/N is m-S-complemented in G/N, then B is m-S-complemented in G.*

Proof. (1) Let T be a subgroup of G such that $AT = G$ and $A \cap T \leq S \leq A$ for some generalized S-quasinormal subgroup S of G. Then $S = \langle L, T \rangle$, where L is a modular and T is an S-quasinormal subgroups of G. Then $NT \cap NA = (T \cap A)N$. Indeed, if $N \leq A$, then $NT \cap NA = NT \cap A = N(T \cap A)$. On the other hand, if $(|A|, |N|) = 1$, then from $AT = G$ we get that $N \leq T$ and so $NT \cap NA = T \cap AN = N(T \cap A)$. Therefore $G/N = (AN/N)(TN/N)$ and $(AN/N) \cap (TN/N) = (AN \cap TN/N) = (A \cap T)N/N \leq SN/N$, where SN/N is a generalized S-quasinormal subgroup of G/N by Lemma 2.2(1). Hence AN/N is m-S-supplemented in G/N.

(2) $B = A(B \cap T)$ and $(B \cap T) \cap A = T \cap A \leq S \leq A$, where S is m-S-permutable in B by Lemma 2.2(2). Hence A is m-S-complemented in B.

(3) See the proof of (1) and use Lemma 2.2(3). □

Lemma 2.4 (See Theorem 5.2.5 in [**23**]). *If H is a modular subgroup of G, then H^G/H_G is hypercyclically embedded in G.*

Recall that $F^*(G)$ is the largest normal quasinilpotent subgroup of G.

Lemma 2.5 (See Theorem 1.2 in [**26**]). *If E is a normal subgroup of G and $F^*(E)$ is hypercyclically embedded in G, then E is hypercyclically embedded in G.*

Recall that formation $\mathfrak{F}$ is a homomorph of groups such that each group G has the smallest normal subgroup (denoted by $G^{\mathfrak{F}}$) whose quotient is still in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if $G \in \mathfrak{F}$ for any group G with $G/\Phi(G) \in \mathfrak{F}$.

Lemma 2.6 (See Lemma 2.16 in [**25**]). *Suppose that $G/N \in \mathfrak{F}$, where $\mathfrak{F}$ is a saturated formation containing all supersoluble groups. If N is hypercyclically embedded in G, then $G \in \mathfrak{F}$.*

Lemma 2.7 (See Lemma 2.10 in [**24**]). *Let P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every maximal subgroup of P has a p-nilpotent supplement in G, then G is p-nilpotent.*

3 - Proofs of Theorems 1.3, 1.4 and 1.5

Proof of Theorem 1.3. Suppose that this theorem is false and consider a counterexample (G, E) for which $|G| + |E|$ is minimal. Let $Z = Z_{\mathfrak{U}}(G)$.

(1) *If R is a normal p'-subgroup of G, then the hypothesis holds for $(G/R, ER/R)$.*

First note that $PR/R \simeq P$ is a Sylow p-subgroup of ER/R, and if V/R is a subgroup of PR/R, then for a Sylow p-subgroup W of V we have $V/R = WR/R$. Moreover, if V/R is a maximal subgroup of PR/R, then $|P : W| = p$ and so W is a maximal subgroup of P. Hence the hypothesis holds for $(G/R, ER/R)$ by Lemma 2.3(1).

(2) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$, and let R be a minimal normal subgroup of G contained in $O_{p'}(G)$. Then the hypothesis holds for $(G/R, ER/R)$ by Claim (1), so $(ER/R)/O_{p'}(ER/R)$ is hypercyclically embedded in G/R and $ER/R \simeq E/E \cap R$ is p-nilpotent. Hence E is p-nilpotent and from

$$(ER/R)/O_{p'}(ER/R) = (ER/R)/(O_{p'}(ER)/R) = (ER/R)/(O_{p'}(E)R/R)$$

and from the G-isomorphisms

$$\begin{aligned}(ER/R)/(O_{p'}(E)R/R) &\simeq ER/O_{p'}(E)R \simeq E/E \cap O_{p'}(E)R \\ &= E/O_{p'}(E)(E \cap R) = E/O_{p'}(E)\end{aligned}$$

we get that $E/O_{p'}(E)$ is hypercyclically embedded in G, contrary to the choice of (G, E). Hence we have (2).

(3) $Z \cap E \leq Z_\infty(E)$.

Since Z is clearly supersoluble, a Sylow q-subgroup Q of Z, where q is the largest prime dividing $|Z|$, is normal and so characteristic in Z. Then Q is normal in G, which implies that $Z = Q$ by Claim (2), so $Z \cap E \leq Z_\infty(E)$ since $(p-1, |E|) = 1$.

(4) *E is p-nilpotent.*

Assume that this is false.

(a) $E = G$.

Since the hypothesis holds for (E, E) by Lemma 2.3(2), in the case when $E \neq G$, the subgroup E is p-nilpotent by the choice of G.

(b) $O_p(G) \neq 1$.

Assume that $O_p(G) = 1$. Lemma 2.7 implies that some maximal subgroup V of P has no a p-nilpotent supplement in G, so V is generalized S-quasinormal in G. Then there are an generalized S-quasinormal subgroup S and a subgroup T of G such that $G = VT$ and $V \cap T \leq S \leq V$. Let A be a modular subgroup and B a S-quasinormal subgroup of G such that $S = \langle A, B\rangle$. Then $BP^x = P^xB = P^x$ for all $x \in G$, so $B \leq P_G = O_p(G) = 1$. Hence $S = A$ and $A_G = 1$, therefore $S \leq Z \leq Z_\infty(G)$ by Lemma 2.4 and Claim (3) since $E = G$ by Claim (a). Since $Z_\infty(G)$ is nilpotent, a Sylow p-subgroup of $Z_\infty(G)$ is normal in G, so $A = S = 1$ since $V_G = 1$. Therefore T is a complement to V in G, so for a Sylow p-subgroup T_p of T we have $|T_p| = p$. Therefore T is p-nilpotent since $(p-1, |E|) = (p-1, |G|) = 1$. Hence every maximal subgroup V of P has a p-nilpotent complement in G, so G is p-nilpotent by Lemma 2.7. This contradiction shows that we have (b).

Let R be a minimal normal subgroup of G contained in $O_p(G)$. First we show that $R \neq P$. Assume that $R = P$ and let V be any maximal subgroup of P. There are an m-S-permutable subgroup S and a subgroup T of G such that $G = VT$ and $V \cap T \leq S \leq V$. Let A be a modular subgroup and B an S-quasinormal subgroup of G such that $S = \langle A, B\rangle$. Then $A_G = 1$, so $A^G \leq Z$ by Lemma 2.4. Therefore $A = 1$ and so $S = B$ is S-quasinormal in G. But then S is normal in G by Lemma 1.2.16 in [**5**]. Hence $S = 1$ and so $T \cap V = 1$. But then $1 < T \cap R < R$, where $T \cap R$ is normal in G. This contradiction shows that $R \neq P$. Therefore the hypothesis holds for G/R by Lemma 2.3(1), so G/R is p-nilpotent by the choice of G. Hence G is p-soluble. Therefore every minimal normal subgroup R of G is a p-group by Claim (2), Hence R is the unique minimal normal subgroup of G and $R \nleq \Phi(G)$, so $R = C_G(R) = O_p(G)$ by [**9**, Ch. A, 15.6]. Hence $R \nleq \Phi(P)$ by [**15**, III, Hilfsatz 3.3]. It is clear also that $|R| > p$, so $Z = 1$.

Final contradiction for (4).

Let V be any maximal subgroup of G. We show that V has a p-nilpotent supplement in G. Assume that this is false. Then the subgroup V is generalized S-quasinormal in G by hypothesis.

First suppose that $R \nleq V$. Then $W = V \cap R$ is normal in P, $|R : W| = p$ and $V_G = 1$. There are a generalized S-quasinormal subgroup S and a subgroup T of G such that $G = VT$ and $V \cap T \leq S \leq V$. Arguing as above, we can show that S is S-quasinormal in G. It follows that $S \leq O_p(G) = R$. Hence $S \leq R \cap V = W$ and so $S^G = S^{PO^p(G)} = S^P \leq W$ by [**5**, Lemma 1.2.16], which implies that $S = 1$. Then T is a complement to V in G, so T is p-nilpotent.

Now let V be any maximal subgroup of P containing R, and let M be a maximal subgroup of G such that $G = R \rtimes M$. Then $M \simeq G/R$ is p-nilpotent, so M is a p-nilpotent supplement to V in G. Thus every maximal subgroup of P has a p-nilpotent supplement in G. Therefore G is p-nilpotent by Lemma 2.7. This contradiction shows that we have (4).

The final contradiction. Claims (2) and (4) imply that $E = P$ is a normal p-subgroup of G. Let R be a minimal normal subgroup of G contained in P.

Then the hypothesis holds for $(G/R, P/R) = (G/R, E/R)$, so the choice of G implies that P/R is hypercyclically embedded in G. Therefore $R \nleq \Phi(P)$ by [**15**, III, Hilfsatz 3.3(a)]. Hence $\Phi(P) = 1$, so P is elementary abelian p-group. If $|R| = p$, then P is hypercyclically embedded in G by the Jordan-Hölder theorem for the chief series. Hence R is not cyclic. Moreover, R is the unique minimal normal subgroup of G contained in P. Indeed, suppose that for some minimal normal subgroup $N \neq R$ of G we also have $N \leq P$. Then P/N is hypercyclically embedded in G and so from the G-isomorphism $RN/N \simeq R$ we get that $|R| = p$, a contradiction.

Let W be a maximal subgroup of R such that W is normal in a Sylow p-subgroup G_p of G. Then $W \neq 1$.

We show that W is S-quasinormal in G. Let B be a complement of R in P and $H = WB$. Then H is a maximal subgroup of P and $W = H \cap R$. Therefore W is S-quasinormal in G in the case when H is S-quasinormal in G by Lemma 2.1(5). From now on we suppose that H is not S-quasinormal in G.

Suppose that H has a p-nilpotent supplement T in G and let S be the normal p-complement in T. Then $P = P \cap HT = H(P \cap T)$, where $P \cap T$ is normal in G since P is abelian. Moreover, $1 < P \cap T < P$ since G is not p-nilpotent. Therefore $R \leq P \cap T$. Then $[R, S] = 1$, so $G = G_pS \leq N_G(W)$. This contradiction shows that H has no a p-nilpotent supplement T in G and hence H is m-S-complemented in G by hypothesis.

Let S and T be subgroups of G such that S is generalized S-quasinormal in G and for which we have $G = HT$ and $H \cap T \leq S \leq H$. And let $S = AB$, where A is modular and B is S-quasinormal in G. Then $N \nleq H$ and so $A_G = 1$, which implies that A^G is hypercyclically embedded in G by Lemma 2.7. But then $A = 1$ since otherwise $N \leq A^G \cap P$ and so $|N| = p$. Therefore $S = B$ is S-quasinormal in G. Since $T \cap H \leq S \leq H$ and H is not S-quasinormal in G, it follows that $T < G$ and for the normal subgroup $T \cap P$ of G we have $1 < T \cap P$. Then $N \leq T$ and so $N \cap H = N \cap S = W$, which implies that W is S-quasinormal in G by Lemma 2.1(5) But then W is normal in G since $G = G_pO^p(G) \leq N_G(W)$ by [**5**, Lemma 1.2.16] and so $W = 1$. Therefore N is cyclic. This contradiction completes the proof of the result. □

P r o o f o f T h e o r e m 1.4. Suppose that this theorem is false and consider a counterexample (G, E) for which $|G| + |E|$ is minimal. Let p be the smallest prime dividing $|E|$ and let P be a Sylow p-subgroup of E.

Note that the hypothesis holds for (E, E) by Lemma 2.3(2), so E is p-supersoluble by Theorem 1.3 and hence E is p-nilpotent since p is the smallest prime dividing $|E|$. Note also that if X is the $p\prime$- Hall subgroup of E, then X is trivial. Indeed, if $X \neq 1$, the hypothesis holds for $(G/X, E/X)$ and for (G, X)

by Lemma 2.3(1). Since obviously $X \neq E$ the choice of G implies that G/X and X are hypercyclically embedded in G. Hence E is hypercyclically embedded in G by the Jordan-Hölder theorem for the chief series. This contradiction shows that $E = P$, so E is hypercyclically embedded in G by Theorem 1.3. □

Proof of Theorem 1.5. This theorem is a corollary of Theorem 1.4, Lemma 2.5 and Lemma 2.6. □

4 - Some applications of the results

Corollary 4.1 (Srinivasan [**29**]). *If the maximal subgroups of the Sylow subgroups of G are S-quasinormal in G, then G is supersoluble.*

Corollary 4.2 (Asaad [**2**]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E. If $G/E \in \mathfrak{F}$ and every maximal subgroup of every Sylow subgroup of E is S-quasinormal in G, then $G \in \mathfrak{F}$.*

Corollary 4.3 (Guo, Shum, Skiba [**11**]). *If the maximal subgroups of the Sylow subgroups of G not having supersoluble supplement in G are normal in G, then G is supersoluble.*

A subgroup H of G is said to be *c-normal* in G [**31**], if there is a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$.

Corollary 4.4 (Wang [**31**]). *If the maximal subgroups of the Sylow subgroups of G are c-normal in G, then G is supersoluble.*

Corollary 4.5 (Alsheik Ahmad [**1**]). *If the maximal subgroups of the Sylow subgroups of G not having supersoluble supplement in G are c-normal in G, then G is supersoluble.*

Corollary 4.6 (Ramadan [**22**]). *Let G be a soluble group. If all maximal subgroups of the Sylow subgroups of $F(E)$ are normal in G, then G is supersoluble.*

Corollary 4.7 (Li, Guo [**19**]). *Let G be a group and E a soluble normal subgroup of G with supersoluble quotient G/E. If all maximal subgroups of the Sylow subgroups of $F(E)$ are c-normal in G, then G is supersoluble.*

Corollary 4.8 (Wey [**33**]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a soluble normal subgroup E such that $G/E \in \mathfrak{F}$. If all maximal subgroups of the Sylow subgroups of $F(E)$ are c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 4.9 (Wei, Wang, Li [**34**]). *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If all maximal subgroups of the Sylow subgroups of $F^*(E)$ are c-normal in G, then $G \in \mathcal{F}$.*

Corollary 4.10 (Asaad, Ramadan, Shaalan [**4**]). *Let G be a group and E a soluble normal subgroup of G with supersoluble quotient G/E. Suppose that all maximal subgroups of any Sylow subgroup of $F(E)$ are S-quasinormal in G. Then G is supersoluble.*

Corollary 4.11 (Li, Wang [**20**]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If all maximal subgroups of $F^*(E)$ are S-quasinormal in G, then $G \in \mathfrak{F}$.*

Corollary 4.12 (Li, Wang [**20**]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If every maximal subgroup of every Sylow subgroup of $F^*(E)$ is S-quasinormal in G, then $G \in \mathfrak{F}$.*

In the paper [**21**] the following concept was introduced: A subgroup H of a group G is said to be $\mathfrak{U}$-s-supplemented in G if G has a subgroup T such that $G = HT$ and $T/T\cap H_G$ is supersoluble. Now let T_0 be a minimal supplement to $T\cap H_G$ in T. Then $T\cap H_G\cap T_0 \leq \Phi(T_0)$ and $T/T\cap H_G = T_0(T\cap H_G)/(T\cap H_G) \simeq T_0/(T_0 \cap (T \cap H_G)$ is supersoluble. Hence T_0 is supersoluble. Therefore H has a supersoluble supplement in G since $G = HT = HT_0$. Therefore we get from Theorem 1.4 the following

Corollary 4.13 (Miao, Guo [**21**]). *Let G be a group and E a soluble normal subgroup of G with supersoluble quotient G/E. If all maximal subgroups of the Sylow subgroups of E are $\mathfrak{U}$-s-supplemented in G, then G is supersoluble.*

Corollary 4.14 (Wei [**33**]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If every maximal subgroup of every Sylow subgroup of E is c-normal in G, then $G \in \mathfrak{F}$.*

Corollary 4.15 (Wei, Wang and Li [**32**]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If every maximal subgroup of every Sylow subgroup of $F^*(E)$ is c-supplemented in G, then $G \in \mathfrak{F}$.*

Corollary 4.16 (Ballester-Bolinches and Guo [**7**]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If every maximal subgroup of every Sylow subgroup of E is c-supplemented in G, then $G \in \mathfrak{F}$.*

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