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Some elementary remarks on lci algebraic cycles

Abstract. In this short note, we collect some known results about representing algebraic cycles by various kind of "nice" (e.g. smooth, local complete intersection, products of local complete intersection) algebraic cycles, up to rational equivalence. We also add a few observations on these representation problems that we were not able to locate in the literature.

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1 - Introduction

The question whether any algebraic cycle on a complex smooth projective or quasi-projective variety X is rationally equivalent to (a multiple of) an algebraic cycle belonging to some special and nice class, is an old problem. To our knowledge, it was first addressed by Hironaka [**Hi**] and Kleiman [**Kl**] who considered the class of *smooth* cycles, by analogy with similar problems in topology originating form the Steenrod problem and subsequent work and questions by Borel-Haefliger. Their results, together with another relevant result by Hartshorne-Rees-Thomas [**HRT**], are briefly reviewed in Section 2.1 below. One might ask the same questions for larger classes of algebraic cycles models. A natural choice, from the standpoint of derived algebraic geometry, is the class of *local complete intersection* (lci) cycles. We collect below a few elementary results for this class of models, and more generally we discuss and compare three different kinds of models: smooth, lci, and products of lci cycles.

These notes aims to recollect some of these representation problems for algebraic cycles, which are fascinating and a bit neglected in the current research panorama.

Most, if not all, of the facts below hold for a complex quasi-projective smooth scheme (and often, over an arbitrary algebraically closed field). However, we will stick to smooth complex projective varieties because these are our main objects of interest in this note.

1.1 - *Acknowledgments*

After writing these notes, we discovered an interesting MathOverflow page (see http://mathoverflow.net/questions/60434) that discusses questions similar to the ones that we address in Section 2.2. Though few details are given, we presume that Sasha's idea on the proof of Theorem 2.1 must have been very similar to ours, and he should be credited for this. The rest of the arguments in the same MO page, especially the one claiming that every Chern class (not only top ones) is, in fact, an lci cycle, are more obscure to us. As Burt Totaro pointed out to us, determinantal varieties are typically not lci, so the argument in the cite MO page is probably incomplete.

We wish to thank Burt Totaro for several useful discussions related to the subject matter of this note, that revived our interest in representation problems for algebraic cycles and corrected some of our mistakes. We also thank Mauro Porta for many inspiring conversations and some joint elementary experiments about the existence of a derived version of the Chow group.

1.2 - *Notations*

In this paper, X will be a complex smooth projective variety, and we will write $\mathrm{CH}_*(X)$ (respectively $\mathrm{CH}^*(X)$) for the Chow group (respectively, the Chow ring) of algebraic cycles modulo rational equivalence on X, *with* $\mathbb{Q}$-*coefficients*. The word *variety* is used as in Fulton's [**Fu**] to mean a reduced, irreducible scheme of finite type over the base field (which will be $\mathbb{C}$ in this note). For $Z \subset X$ a closed subscheme, $[Z]$ will denote the algebraic cycle (modulo rational equivalence) associated to Z, as in [**Fu**, 1.5].

2 - Smooth, lci, and plci algebraic cycles

D e f i n i t i o n 2.1. Let X be a smooth complex projective variety.

- An algebraic cycle $\mathfrak{z}$ on X is said to be *lci* or *local complete intersection* (respectively, *smooth*) if there exist closed subschemes $i_j : Z_j \hookrightarrow X$, and $\alpha_j \in \mathbb{Q}$, $j = 1, \ldots, n$, such that each i_j (respectively, each Z_j) is a regular closed immersion[1] (resp. is smooth over $\mathbb{C}$), and $\mathfrak{z}$ is rationally equivalent to the algebraic cycle $\sum_{j=1}^{n} \alpha_j [Z_j]$ (i.e. $[\mathfrak{z}] = \sum_{j=1}^{n} \alpha_j [Z_j]$ in $\mathrm{CH}_*(X)$). We denote by $\mathrm{CH}_*^{\mathrm{lci}}(X)$ (respectively, $\mathrm{CH}_*^{\mathrm{sm}}(X)$) the sub-$\mathbb{Q}$-vector space of $\mathrm{CH}_*(X)$ generated by lci (resp., smooth[2]) cycles.

- An algebraic cycle $\mathfrak{z}$ on X is said to be *plci* (or *product of local complete intersection*) if it is rationally equivalent to a product of lci cycles inside the graded $\mathbb{Q}$-algebra $\mathrm{CH}^*(X)$. We denote by $\mathrm{CH}_{\mathrm{plci}}^*(X)$ the sub-$\mathbb{Q}$-vector space of $\mathrm{CH}^*(X)$ generated by lci cycles (or, equivalently, the sub-$\mathbb{Q}$-algebra of $\mathrm{CH}^*(X)$ generated by lci cycles).

It is natural to formulate the following questions.

C o n j e c t u r e 2.1. $\mathrm{CH}_*^{\mathrm{sm}}(X) = \mathrm{CH}_*(X)$.

C o n j e c t u r e 2.2. $\mathrm{CH}_*^{\mathrm{lci}}(X) = \mathrm{CH}_*(X)$.

C o n j e c t u r e 2.3. $\mathrm{CH}_{\mathrm{plci}}^*(X) = \mathrm{CH}_*(X)$.

We have obvious implications

$$\text{Conjecture 2.1} \Rightarrow \text{Conjecture 2.2} \Rightarrow \text{Conjecture 2.3}.$$

To our knowledge, none of the previous implications is known to be reversible in general.

R e m a r k 2.1. *By Bertini's theorem,* global *complete intersections* (*gci*) *in* $X = \mathbb{P}^n$ *or in a smooth affine variety* $X = \mathrm{Spec}\, A$ *are smoothable, i.e., can be deformed to smooth subschemes. Moreover, the parameter space of the deformation can always be assumed to be a non-singular curve* (*see, e.g., the argument right before* [**Ha2**, *Example 29.0.1*])*. For such* X*, we have* $\mathrm{CH}_{\mathrm{alg},\,*}^{\mathrm{gci}}(X) = \mathrm{CH}_{\mathrm{alg},\,*}^{\mathrm{sm}}(X)$*, for the Chow subgroups of gci and smooth cycles, modulo algebraic equivalence. However, on a general smooth projective* X*, it is not true that an lci subscheme is smoothable: a singular* (-1)*-curve on a smooth surface is rigid, hence is not smoothable. Further examples of non-smoothable lci curves in* $\mathbb{P}^3$ *are given in* [**Ha-Hi**, § 5]*. Since we are interested in algebraic cycles, and not in subschemes, it might nonetheless be true that* $\mathrm{CH}_{\mathrm{alg},\,*}^{\mathrm{lci}}(X) = \mathrm{CH}_{\mathrm{alg},\,*}^{\mathrm{sm}}(X)$*, or that*

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[1]Or, equivalently, an lci morphism.

[2]This definition of smooth algebraic cycles is equivalent to the one considered in [**Hi**, **Kl**, **HRT**], since a smooth $\mathbb{C}$-scheme has a finite number of irreducible components, each one being again smooth over $\mathbb{C}$.

$\mathrm{CH}_*^{\mathrm{lci}}(X) = \mathrm{CH}_*^{\mathrm{sm}}(X)^3$. *For example, it is possible that an arbitrarily given lci subscheme becomes smoothable after adding a suitable smooth subscheme (this is indeed known to be the case in codimension* 1, *e.g. for the* (-1) *curves mentioned above). This remark is essentially due to Burt Totaro, who also provided the reference* [**Ha-Hi**]. *We wish to thank him for his clarifications.*

2.1 - *Discussion of Conjecture 2.1*

Conjecture 2.1 have been addressed by Hironaka (for Chow groups with integer and rational coefficients, [**Hi**]), and by Kleiman (for Chow groups with rational coefficients, [**Kl**]). Kleiman proved that for a connected, smooth (quasi-)projective scheme Y of dimension d over an algebraically closed field, then $\mathrm{CH}_i^{\mathrm{sm}}(Y) = \mathrm{CH}_i(Y)$, for any $i < (d+2)/2$ ([**Kl**, Theorem 5.8]).

Later, Hartshorne-Rees-Thomas proved that Conjecture 2.1 is false *with integer coefficients*, by producing algebraic cycles with integer coefficients on $\mathrm{Grass}_{3,3}$ that are not smoothable even up to homological equivalence ([**HRT**, Theorem 1]). However, the examples in [**HRT**] are not known to contradict Conjecture 2.1, i.e. it is not known whether some integer multiples of them are smoothable.

The next remark shows that, in order to check whether Conjecture 2.1 holds for X, it is enough to test it on all Chern classes of all vector bundles on X.

P r o p o s i t i o n 2.1. *Let X be a smooth complex projective variety. If for all vector bundles E on X, we know that $c_i(E)$ is a smooth cycle, for any $i \geq 0$, then Conjecture 2.1 is true.*

P r o o f. Using the syzygy theorem as in the proof of [**Kl**, Lemma 5.4], we know that, for any closed irreducible codimension p subvariety $Z \subset X$, there is a vector bundle E on X, and an integer n, such that the following equality

$$\pm(p-1)![Z] = c_p(E) - n\mathrm{H}^p$$

holds in $\mathrm{CH}^*(X)$. Here H is the class of a hyperplane section. Hence H^p is smooth for any p, and we conclude.   □

2.2 - *Discussion of Conjecture 2.2*

First of all, as in the case of smooth cycles, in order to prove Conjecture 2.2 for X, it is enough to test it on all Chern classes of vector bundles on X:

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[3]Another reason to believe this comes from the results of Hironaka and Kleiman, see Section 2.1

P r o p o s i t i o n   2.2. *Let X be a smooth complex projective variety. If for all vector bundles E on X, we know that $c_i(E)$ is an lci cycle, for any $i \geq 0$, then Conjecture 2.2 is true.*

P r o o f.   Using the syzygy theorem as in the proof of [**Kl**, Lemma 5.4], we know that, for any closed irreducible codimension p subvariety $Z \subset X$, there is a vector bundle E on X, and an integer n, such that the following equality

$$\pm(p-1)![Z] = c_p(E) - n\mathrm{H}^p$$

holds in $\mathrm{CH}^*(X)$. Here $\mathrm{H} = c_1(\mathcal{O}(1))$ is the class of an hyperplane section, hence H^p is lci for any p (e.g. by Corollary 2.1 below), and we conclude. $\square$

Therefore we can reduce to study the property of being lci for Chern classes of vector bundles on X. For the first Chern classes, it is classical that they all belong to $\mathrm{CH}^{\mathrm{sm}}_{d-1}(X)$, hence to $\mathrm{CH}^{\mathrm{lci}}_{d-1}(X)$, for $d = \dim X$. A further observation in this direction will be a corollary of the following

T h e o r e m   2.1. *Let E be a rank r vector bundle on X, and L be a globally generated line bundle on X, such that $E \otimes L$ is globally generated. Then, the Chern classes $c_1(L)^i c_{r-i}(E \otimes L)$, are lci for all $i = 0, \ldots, r$.*

P r o o f.   Define $x_i := c_1(L)^i c_{r-i}(E \otimes L)$. By developing the Chern class of the tensor product $E \otimes L^{\otimes m+1} = (E \otimes L) \otimes L^{\otimes m}$, we get

$$c_r((E \otimes L) \otimes L^{\otimes m}) = \sum_{i=0}^{r} c_1(L^{\otimes m})^i c_{r-i}(E \otimes L) = \sum_{i=0}^{r} m^i x_i$$

where we have used that $c_1(L^{\otimes m}) = mc_1(L)$. By taking $m = 0, \ldots, r$ we get a system of $(r+1)$ linear equations in the $(r+1)$ unknowns $x_0, \ldots, x_r$ with integer coefficients, and constant terms given by the classes $c_r(E \otimes L^{m+1})$, for $m = 0, \ldots, r$. Note that the top Chern class $c_{top}(F)$ of any globally generated vector bundle F on X is represented by the zero scheme $Z(s)$ of some global section of F, and $Z(s) \hookrightarrow X$ is a regular closed immersion of codimension $\mathrm{rk}(F)$, since X is Cohen-Macaulay (see, e.g., [**Ha1**, Thm. 8.21A (c)]). Hence $c_{top}(F)$ is an lci cycle for any globally generated vector bundle F on X; therefore this is true for $c_r(E \otimes L^{m+1})$, for $m = 0, \ldots, r$, by our hypothesis on L.

The matrix of the corresponding homogeneous system is of the form

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 4 & \ldots & 2^r \\ \vdots & \vdots & & & \vdots \\ 1 & r & r^2 & \ldots & r^r \end{bmatrix}$$

which has a non-vanishing determinant since it is a Vandermonde matrix whose rows are all different. By solving the system, we find $x_0, \ldots, x_r$ expressed as a rational linear combination of the lci classes $c_r(E \otimes L), \ldots, c_r(E \otimes L^{\otimes r+1})$. Thus all the x_i's are lci.[4]   □

Corollary 2.1. *Let E be a vector bundle of rank r on a smooth projective variety X. The top Chern class $c_r(E)$ is lci.*

Proof. Pick a line bundle L on X as in the statement of Theorem 2.1 (it is classical that such a sufficiently ample line bundle exists). Consider the top Chern class of $E \otimes L$; we have

$$c_r(E \otimes L) = \sum_{i=0}^{r} c_1(L)^i c_{r-i}(E).$$

Thus

$$c_r(E) = c_r(E \otimes L) - \sum_{i=1}^{r} c_1(L) c_{r-i}(E).$$

By Theorem 2.1, the right-hand side of the previous equation is indeed a sum of lci classes.   □

At the moment we are unable to prove that $c_i(E)$ is lci for any vector bundle E on X, and $1 < i < \operatorname{rank} E$ (i.e., we are not able to prove Conjecture 2.2).

Remark 2.2. *Let us compare Corollary 2.1 with the best, up to now, smoothing result for cycles with rational coefficients, i.e. [**Kl**, Theorem 5.8]. Let $d := \dim X$. As already recalled above, Kleiman shows that $\mathrm{CH}_i(X) = \mathrm{CH}_i^{\mathrm{sm}}(X)$ for $i < (d+2)/2$. This result is much stronger (and effective) than Corollary 2.1, for $\operatorname{rank} E > (d-2)/2$. However, for $2 \leq \operatorname{rk} E \leq (d-2)/2$, [**Kl**, Theorem 5.8] does not cover Corollary 2.1.*

2.3 - *Discussion of Conjecture 2.3*

Conjecture 2.3 is the weakest one among the three we have listed. We remark the following

Proposition 2.3. *If $c_i(E)$ is plci, for any vector bundle E over X, and for any $i \geq 0$, then Conjecture 2.3 is true.*

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[4]By observing that $x_0 = c_r(E \otimes L)$ and $x_r = c_1(L)^r$ are lci, we could reduce ourselves to consider a smaller linear system of $r-1$ equations in $r-1$ unknowns. However, this does not yield any significant simplification in our argument.

P r o o f. Just recall that the Chern character Ch : $K_0(X) \otimes \mathbb{Q} \to \mathrm{CH}^*(X)$ is an isomorphism of $\mathbb{Q}$-algebras, since X is smooth, and that the Chern character is a polynomial with rational coefficients in the Chern classes.   □

Suppose we know that Conjecture 2.3 holds. Then it would be tempting to deduce Conjecture 2.2 by a deformation argument saying that the intersection of two lci cycles is rationally equivalent to an lci cycle. To do this, one might use (at least) two strategies: deformation to the normal cone or a moving lemma. We have not fully explored the first approach, but we are a bit skeptical about the second one because the classical moving lemma deforms cycles, not specified subvarieties (or rather, it deforms a specified subvariety into a full cycle). So it is not obvious to us how a moving lemma argument might be combined with a deformation-theoretic argument on the moduli space of lci subschemes in X, in order to prove that the product of any two lci cycles is indeed an lci cycle (i.e. that plci $\Rightarrow$ lci).

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