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The Levi decomposition of a graded Lie algebra

Abstract. We show that a graded Lie algebra admits a Levi decomposition that is compatible with the grading.

Keywords. Levi decomposition, graded Lie algebra.

Mathematics Subject Classification (2010): Primary: 17B05, secondary: 17B70.

1 - introduction

We assume throughout that Lie algebras are real and finite-dimensional. In 1905, E. E. Levi [3] showed that every Lie algebra may be decomposed as a direct sum:

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{r},$$

where \mathfrak{r} is the radical, the maximal solvable ideal of \mathfrak{g} , and \mathfrak{l} is a semisimple subalgebra of \mathfrak{g} . This is one of the foundations of Lie theory.

In more recent times, \mathbb{Z} -graded Lie algebras have come to assume an important role: for such an algebra \mathfrak{g} , we may write $\mathfrak{g} = \sum_{n \in \mathbb{Z}} \mathfrak{g}_n$, where $\mathfrak{g}_n = \{0\}$ for all but finitely many n and $[\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \mathfrak{g}_{m+n}$ for all m and n . Given a grading, there is an associated derivation, δ_1 say, which is determined by linearity and the condition that $\delta_1 X = nX$ for all $X \in \mathfrak{g}_n$. Conversely, given a diagonalisable derivation δ_1 , all of whose eigenvalues are integers, we obtain a grading by defining \mathfrak{g}_n to be the eigenspace for the eigenvalue n when n is an eigenvalue, and $\{0\}$ otherwise.

To the best of our knowledge, the interplay between the grading of an algebra and the Levi decomposition has not been made explicit, and this paper

Received: December 2, 2018; accepted in revised form: January 15, 2019.

This work was developed while the second author was supported by a Visiting Scientist Grant from the Università di Padova

fills this gap. Suppose that \mathfrak{g} is a \mathbb{Z} -graded Lie algebra. The radical \mathfrak{r} is a characteristic ideal of \mathfrak{g} , so $\delta_1\mathfrak{r} \subseteq \mathfrak{r}$, whence $\mathfrak{r} = \sum_n \mathfrak{r}_n$, where $\mathfrak{r}_n = \mathfrak{g}_n \cap \mathfrak{r}$. But the same need not hold for a generic Levi subalgebra \mathfrak{l} . We will show how to choose a δ_1 -invariant Levi subalgebra \mathfrak{l} ; for this choice of \mathfrak{l} it follows immediately that $\mathfrak{l} = \sum_n \mathfrak{l}_n$, where $\mathfrak{l}_n = \mathfrak{g}_n \cap \mathfrak{l}$.

More generally, Lie algebras may be graded over \mathbb{Z}^d , where $d > 1$; root space decompositions of semisimple Lie algebras are examples of this. These gradings correspond to commuting families of diagonalisable derivations with integer eigenvalues.

We consider a slightly more general structure: an abelian Lie algebra \mathfrak{a} and a homomorphism $\delta : H \mapsto \delta_H$ from \mathfrak{a} into the Lie algebra of derivations of the Lie algebra \mathfrak{g} ; we assume that each δ_H is semisimple, but not necessarily diagonalisable over \mathbb{R} . Here is our main result.

Theorem 1.1. *Let $\{\delta_H : H \in \mathfrak{a}\}$ be a commuting family of semisimple derivations of a Lie algebra \mathfrak{g} , and let \mathfrak{r} be the radical of \mathfrak{g} . Then there exists a Levi subalgebra \mathfrak{l} such that $\delta_{\mathfrak{a}}\mathfrak{l} \subseteq \mathfrak{l}$.*

There are a number of arguments in the literature that this theorem illuminates and simplifies: see, for example, [1] and [4].

2 - Proof of Theorem 1.1

We follow the standard proof of the Levi decomposition, all the while keeping an eye on the derivations δ_H . First of all, we suppose that the derivations are inner, that is, $\mathfrak{a} \subset \mathfrak{g}$ and $\delta_H = \text{ad}(H)$ for each $H \in \mathfrak{a}$. Then $[\mathfrak{g}, \mathfrak{r}]$ and $[\mathfrak{r}, \mathfrak{r}]$ are $\text{ad}(\mathfrak{a})$ -invariant ideals in \mathfrak{g} that are contained in \mathfrak{r} . We consider several cases.

Case 1: There is an $\text{ad}(\mathfrak{a})$ -invariant ideal \mathfrak{i} such that $\{0\} \subset \mathfrak{i} \subset \mathfrak{r}$.

In this case, we argue by induction on dimension. The derivations δ_H induce derivations on $\mathfrak{g}/\mathfrak{i}$, and $\mathfrak{r}/\mathfrak{i}$ is the radical of $\mathfrak{g}/\mathfrak{i}$, so we may write $\mathfrak{g}/\mathfrak{i} = \mathfrak{h}/\mathfrak{i} \oplus \mathfrak{r}/\mathfrak{i}$, where \mathfrak{h} contains \mathfrak{i} and is $\text{ad}(\mathfrak{a})$ -invariant, and $\mathfrak{h}/\mathfrak{i}$ is semisimple. Then \mathfrak{i} is the radical of \mathfrak{h} , and by induction, we may write \mathfrak{h} as $\mathfrak{l} \oplus \mathfrak{i}$, where \mathfrak{l} is semisimple and $\text{ad}(\mathfrak{a})$ -invariant. Then $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{r}$ is an $\text{ad}(\mathfrak{a})$ -invariant Levi decomposition of \mathfrak{g} , and the result is established in Case 1.

Since $[\mathfrak{g}, \mathfrak{r}]$ is an $\text{ad}(\mathfrak{a})$ -invariant ideal in \mathfrak{g} , if we are not in Case 1, then either $[\mathfrak{g}, \mathfrak{r}] = \{0\}$ or $[\mathfrak{g}, \mathfrak{r}] = \mathfrak{r}$. Similarly, either $[\mathfrak{r}, \mathfrak{r}] = \{0\}$ or $[\mathfrak{r}, \mathfrak{r}] = \mathfrak{r}$; this latter case cannot occur as \mathfrak{r} is solvable.

Case 2a: $[\mathfrak{g}, \mathfrak{r}] = \{0\}$.

In this case, \mathfrak{r} is the centre of \mathfrak{g} , and $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{r}$, where $\mathfrak{l} = [\mathfrak{g}, \mathfrak{g}]$; both summands are $\text{ad}(\mathfrak{a})$ -invariant and \mathfrak{l} is a Levi subalgebra.

Case 2b: $[\mathfrak{g}, \mathfrak{r}] = \mathfrak{r}$.

In this case, $[\mathfrak{r}, \mathfrak{r}] = \{0\}$, so $\text{ad}(X)^2 = 0$ for all $X \in \mathfrak{r}$; moreover, the centre of \mathfrak{g} is trivial, for otherwise we are in case 1. We take a Levi decomposition $\mathfrak{l} \oplus \mathfrak{r}$ of \mathfrak{g} , and modify \mathfrak{l} to achieve the desired decomposition.

Take $H \in \mathfrak{a} \setminus \{0\}$, and write $H = H_{\mathfrak{l}} + H_{\mathfrak{r}}$, where $H_{\mathfrak{l}} \in \mathfrak{l}$ and $H_{\mathfrak{r}} \in \mathfrak{r}$. Since $\text{ad}(H)$ maps \mathfrak{r} into \mathfrak{r} and is semisimple, $\mathfrak{r} = \ker \text{ad}(H) \oplus \text{range ad}(H)$, so we may write $H_{\mathfrak{r}}$ as $H_0 + \text{ad}(H)X$, where $H_0 \in \ker(\text{ad}(H))$ and $X \in \mathfrak{r}$.

Now $\exp(\text{ad}(X))$ is an automorphism of \mathfrak{g} , so we may define a new Levi factor, $\tilde{\mathfrak{l}}$ say, to be $\exp(\text{ad}(X))\mathfrak{l}$; we also define \tilde{H} and $\tilde{H}_{\mathfrak{l}}$ to be $\exp(\text{ad}(X))H$ and $\exp(\text{ad}(X))H_{\mathfrak{l}}$. Now $\text{ad}(X)H_{\mathfrak{l}} = -[H_{\mathfrak{l}}, X] = -[H, X]$ as $[H_{\mathfrak{r}}, X] = 0$, whence

$$\tilde{H}_{\mathfrak{l}} = H_{\mathfrak{l}} + \text{ad}(X)H_{\mathfrak{l}} = H_{\mathfrak{l}} - [H, X] = H - H_{\mathfrak{r}} - [H, X] = H - H_0.$$

Hence $H = \tilde{H}_{\mathfrak{l}} + H_0$.

By definition, $[H, H_0] = 0$, so $[H, \tilde{H}_{\mathfrak{l}}] = 0$. Moreover, $\text{ad}(H)$ is semisimple by definition. Further, the action of $\text{ad}(\tilde{H}_{\mathfrak{l}})$ on $\tilde{\mathfrak{l}}$ coincides with the quotient action of $\text{ad}(\tilde{H})$ on $\tilde{\mathfrak{l}} \oplus \mathfrak{r}/\mathfrak{r}$, which is semisimple by definition, whence the action of $\text{ad}(\tilde{H}_{\mathfrak{l}})$ is also semisimple on \mathfrak{g} (see, for example, [2, Corollary C.18, p. 483]). As $\text{ad}(H)$ and $\text{ad}(\tilde{H}_{\mathfrak{l}})$ commute and are both semisimple, $\text{ad}(H - \tilde{H}_{\mathfrak{l}})$ is also semisimple. However, $\text{ad}(H - \tilde{H}_{\mathfrak{l}}) = \text{ad}(H_0)$ and $\text{ad}(H_0)$ is nilpotent as $H_0 \in \mathfrak{r}$. We deduce that $\text{ad}(H_0)$ is trivial. As the centre of \mathfrak{g} is trivial, $H_0 = 0$. In conclusion, $H = \tilde{H}_{\mathfrak{l}} \in \tilde{\mathfrak{l}}$, and $\tilde{\mathfrak{l}}$ is $\text{ad}(H)$ -invariant.

This argument shows that we can take a Levi subalgebra \mathfrak{l} that is $\text{ad}(H)$ -invariant for a given H in \mathfrak{a} , but \mathfrak{a} may not be 1-dimensional, so more is required. After passing to the complexification if necessary, we may suppose that $\mathfrak{r} = \sum_{\alpha \in \Sigma} \mathfrak{r}_{\alpha}$, where Σ is a finite subset of \mathfrak{a}^* , and $[H, X] = \alpha(H)X$ for all $X \in \mathfrak{r}_{\alpha}$ and all $H \in \mathfrak{a}$. We take $H \in \mathfrak{a}$ such that $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$, and assume that $H \in \mathfrak{l}$ so that \mathfrak{l} is $\text{ad}(H)$ -invariant.

Now if $H' \in \mathfrak{a}$, and we write $H' = H'_{\mathfrak{l}} + H'_{\mathfrak{r}}$, where $H'_{\mathfrak{l}} \in \mathfrak{l}$ and $H'_{\mathfrak{r}} \in \mathfrak{r}$, then H and H' commute by definition, and $H_{\mathfrak{r}}$ and $H'_{\mathfrak{r}}$ commute because \mathfrak{r} is abelian. Further, using the identification of \mathfrak{l} with $\mathfrak{g}/\mathfrak{r}$, we see that $\text{ad}(H)$ and $\text{ad}(H')$ induce commuting derivations of \mathfrak{l} , which may be identified with $\text{ad}(H_{\mathfrak{l}})$ and $\text{ad}(H'_{\mathfrak{l}})$, whence $H_{\mathfrak{l}}$ and $H'_{\mathfrak{l}}$ commute. Write $H_{\mathfrak{r}}$ as $\sum_{\alpha} H_{\alpha}$ and $H'_{\mathfrak{r}}$ as $\sum_{\alpha} H'_{\alpha}$, where $H_{\alpha}, H'_{\alpha} \in \mathfrak{r}_{\alpha}$. Then

$$\begin{aligned} 0 &= [H, H'] = [H_{\mathfrak{l}} + H_{\mathfrak{r}}, H'_{\mathfrak{l}} + H'_{\mathfrak{r}}] = [H_{\mathfrak{l}}, H'_{\mathfrak{l}}] + [H_{\mathfrak{r}}, H'_{\mathfrak{l}}] \\ &= [H, H'_{\mathfrak{l}}] - [H', H_{\mathfrak{r}}] = \sum_{\alpha} \alpha(H)H'_{\alpha} - \sum_{\alpha} \alpha(H')H_{\alpha}. \end{aligned}$$

Since the “root space” decomposition of \mathfrak{r} is a direct sum, $\alpha(H)H'_{\alpha} = \alpha(H')H_{\alpha}$. But $H_{\alpha} = 0$ for all α , since $H \in \mathfrak{l}$, whence $H'_{\alpha} = 0$ unless $\alpha = 0$. Finally, much

as argued above, $\text{ad}(H'_0)$ is both semisimple and nilpotent, hence null, whence $H'_0 = 0$, and $H' \in \mathfrak{l}$, as required.

We conclude our discussion of Cases 1 to 2b by affirming that Theorem 1 holds when all the derivations δ_H are inner.

It remains to discuss the general case, where some or all of the derivations are not inner. In this case, we define \mathfrak{g}_1 to be the vector space $\mathfrak{g} \oplus \mathfrak{a}$, and consider \mathfrak{g} and \mathfrak{a} as subspaces of \mathfrak{g}_1 in the usual way. We take the Lie product $[\cdot, \cdot]_1$ on \mathfrak{g}_1 that is determined by linearity, antisymmetry, and the requirements that

$$[H, H'] = 0, \quad [X, X']_1 = [X, X'] \quad \text{and} \quad [H, X]_1 = \delta_H X$$

for all $H, H' \in \mathfrak{a}$ and all $X, X' \in \mathfrak{g}$.

Now there exist a semisimple subalgebra \mathfrak{l}_1 and a solvable ideal \mathfrak{r}_1 , both $\text{ad}(\mathfrak{a})$ -invariant, such that $\mathfrak{g}_1 = \mathfrak{l}_1 \oplus \mathfrak{r}_1$. By construction, $\mathfrak{l}_1 = [\mathfrak{l}_1, \mathfrak{l}_1] \subseteq [\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}$, so \mathfrak{l}_1 is a $\delta_{\mathfrak{a}}$ -invariant semisimple subalgebra of \mathfrak{g} . Further, it is easy to see that $\mathfrak{g} \cap \mathfrak{r}_1 = \mathfrak{r}$. Finally, since $\mathfrak{g}_1 = \mathfrak{l}_1 \oplus \mathfrak{r}_1$ and $\mathfrak{l}_1 \subseteq \mathfrak{g}$, it follows that

$$\mathfrak{g} = \mathfrak{l}_1 \oplus (\mathfrak{g} \cap \mathfrak{r}_1) = \mathfrak{l}_1 \oplus \mathfrak{r},$$

where both summands are $\delta_{\mathfrak{a}}$ -invariant, as required.

3 - Two corollaries

The first corollary is immediate.

Corollary 3.1. *Suppose that \mathfrak{g} is a \mathbb{Z}^d -graded Lie algebra, so that $\mathfrak{g} = \sum_m \mathfrak{g}_m$, where $[\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \mathfrak{g}_{m+n}$ for all $m, n \in \mathbb{Z}^d$. Then $\mathfrak{r} = \sum_m \mathfrak{r}_m$, where $\mathfrak{r}_m = \mathfrak{r} \cap \mathfrak{g}_m$, and there is a Levi subalgebra \mathfrak{l} such that $\mathfrak{l} = \sum_m \mathfrak{l}_m$, where $\mathfrak{l}_m = \mathfrak{l} \cap \mathfrak{g}_m$.*

Our decomposition also provides us with some information about algebras of derivations.

Corollary 3.2. *Suppose that $\delta_{\mathfrak{a}}$ is an abelian algebra of semisimple derivations of \mathfrak{g} , and that $\mathfrak{l} \oplus \mathfrak{r}$ is a Levi decomposition of \mathfrak{g} into $\delta_{\mathfrak{a}}$ -invariant summands. Then there are commuting algebras $\delta_{\mathfrak{a}, \mathfrak{l}}$ and $\delta_{\mathfrak{a}, \mathfrak{r}}$ of commuting semisimple derivations of \mathfrak{g} that preserve \mathfrak{l} and \mathfrak{r} such that $\delta_{\mathfrak{a}} \subseteq \delta_{\mathfrak{a}, \mathfrak{l}} \oplus \delta_{\mathfrak{a}, \mathfrak{r}}$; further, $\delta_{\mathfrak{a}, \mathfrak{l}}$ may be identified with $\text{ad}(\mathfrak{b})$, where \mathfrak{b} is a commutative subalgebra of \mathfrak{l} , and every element of $\delta_{\mathfrak{a}, \mathfrak{r}}$ annihilates \mathfrak{l} .*

To see this, recall that each derivation δ_H of \mathfrak{g} induces a derivation of $\mathfrak{l} \oplus \mathfrak{r}/\mathfrak{r}$, which we may identify with an inner derivation $\text{ad}(H_{\mathfrak{l}})$ of \mathfrak{l} ; take $\delta_{\mathfrak{a}, \mathfrak{l}}$ to be the algebra of all derivations of \mathfrak{g} of the form $\text{ad}(H_{\mathfrak{l}})$; these commute by

construction. Define $\delta_{H,\tau}$ to be the derivation $\delta_H - \text{ad}(H_{\mathfrak{l}})$; then $\delta_{H,\tau}(X) = 0$ for all $X \in \mathfrak{l}$. It follows that $[\delta_{H,\tau}, \text{ad}(X)] = 0$ for all $X \in \mathfrak{l}$, and so

$$[\delta_{H,\tau}, \delta_{H',\tau}] = [\text{ad}(H_{\mathfrak{l}}) + \delta_{H,\tau}, \text{ad}(H'_{\mathfrak{l}}) + \delta_{H',\tau}] = [\delta_H, \delta_{H'}] = 0.$$

In conclusion, the various $\delta_{H,\tau}$ commute amongst themselves and with the $\text{ad}(H_{\mathfrak{l}})$, which also commute amongst themselves. The corollary follows.

4 - Afterword

After a preliminary version of this note was circulated, it was pointed out to us that our result follows from some already published results. However, in each case, the previous proof is considerably more technical than ours and require more background in Lie theory or algebra.

Ironically, although the Levi decomposition appears in many textbooks and research papers, at the time of writing of this article, Levi's paper has been cited 5 times, according to *Zentralblatt für Mathematik*; so much for the existence of a correlation between citation numbers and significance of a contribution.

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