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Azat Miftakhov

Modulus of continuity for a martingale sequence

Abstract. Given a martingale sequence of random fields that satisfies a natural assumption of boundedness, it is shown that the pointwise limit of this sequence can be modified in such a way that a certain class of moduli of continuity is preserved. That is, if every element of the sequence admits a given modulus of continuity, one can construct a modification of the limiting random field so that this new field also admits the same modulus of continuity. Additionally, it is shown that requiring further smoothness and a stronger notion of boundedness for the original sequence guarantees further smoothness of the limiting field and a stronger mode of convergence to this limit. Moreover, the modulus of continuity is also preserved for the derivatives.

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For a scalar-valued martingale, the classical Doob martingale convergence theorem states the almost sure convergence, as time goes to infinity, for a martingale that is bounded in a certain functional-analytic sense (for precise definitions see, e.g., [**1**]). Various generalizations of the Doob convergence theorem exist for martingales whose values are random elements taking values in an infinitely dimensional Banach space. In particular, the Doob convergence theorem holds if the Banach space satisfies the Radon–Nikodym property (e.g., see [**3**, Theorem 2.9] and also [**2**, **4**]).

Informally, the main result of this note states that the modulus of continuity of a martingale taking values in the space of continuous functions is preserved in the limit. Reasoning in a similar way, under minimal technical assumptions, one can prove that the limit of a martingale of entire functions is itself an entire

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function. Observe, however, that the order and the type of the entire function need not be preserved in passage to the limit: the simple example

$$e^{e^x} = \sum_{n=0}^{\infty} \frac{e^{nx}}{n!}, \tag{1}$$

shows that without additional assumptions the expectation of a random variable taking values in entire function of order one may itself have infinite order, and, consequently, a martingale sequence of functions of order 1 may converge to an entire function of an infinite order.

Let $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ be a *modulus of continuity* which is continuous, increasing, and subadditive. In other words, θ is a continuous increasing function, $\theta(0) = 0$, that satisfies $\theta(x+y) \leq \theta(x) + \theta(y)$ for all $x, y \in \mathbb{R}_+$. Further we are interested exclusively in continuous, increasing, and subadditive moduli, and for the sake of brevity they are referred to simply as moduli. A canonical example of a modulus of continuity is given by $\theta(x) = x^\alpha$, $\alpha \in (0, 1]$, which describe the property of Hölder continuity.

For any function f on a compact domain $E \subset \mathbb{R}^d$, we say that it *admits* the modulus of continuity θ if and only if

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{\theta(|x - y|)} < +\infty. \tag{2}$$

We are going to study (2) for random fields that are elements of a martingale sequence. To simplify the technical matters we bound ourselves to considering only discrete-time martingales. The proof can be easily modified to cover the continuous case as well, however one needs to impose additional technical conditions.

Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n \geq 0}, \mathcal{P})$. For the sake of convenience we assume that both the probability space and the filtration are complete. In this setting the following statement holds.

Theorem 1. *For a compact domain $E \subset \mathbb{R}^d$ let $\{(\xi_n(x), x \in E)\}_{n \geq 0}$ be a sequence of random fields such that their realizations admit a modulus of continuity θ almost surely. Set*

$$M_n \stackrel{\text{def}}{=} \sup_x |\xi_n(x)| + \sup_{x \neq y} \frac{|\xi_n(x) - \xi_n(y)|}{\theta(|x - y|)}, \tag{3}$$

and assume that $(\xi_n(x), \{\mathcal{F}_n\}_{n \geq 0})$ is a martingale for every fixed $x \in E$. If

$$\sup_n \mathbb{E}\left[M_n\right] < +\infty, \tag{4}$$

then there exist a random field $(\xi(x), x \in E)$ *such that its realizations admit the modulus of continuity* θ *almost surely and such that*

$$(\xi_n(x), x \in E) \underset{n\to\infty}{\longrightarrow} (\xi(x), x \in E) \tag{5}$$

pointwise almost surely.

Proof. Clearly, $(M_n, \{\mathcal{F}_n\}_{n\geq 0})$ is a submartingale. The condition (4) means that this submartingale is bounded. By the classical Doob's martingale convergence theorem (e.g., see [**1**]) one has that

$$M_n \underset{n\to\infty}{\longrightarrow} M \tag{6}$$

almost surely, where M is a random variable with expectation $\mathbb{E}\,[M] < +\infty$. Condition (4) also implies that the martingale $(\xi_n(x), \{\mathcal{F}_n\}_{n\geq 0})$ is bounded for every fixed $x \in E$. Again, Doob's martingale convergence theorem yields

$$\xi_n(x) \underset{n\to\infty}{\longrightarrow} \tilde{\xi}(x) \tag{7}$$

almost surely, for some random variable $\tilde{\xi}(x)$ with expectation $\mathbb{E}\left[\tilde{\xi}(x)\right] < +\infty$. In this way one can define the random field $(\tilde{\xi}(x), x \in E)$. Note, however, that neither can we claim that realizations of this field admit θ almost surely, nor can we claim $(\xi_n(x), x \in E) \underset{n\to\infty}{\longrightarrow} (\tilde{\xi}(x), x \in E)$ pointwise almost surely. The almost sure convergence merely takes place for every fixed $x \in E$, and the exceptional set of zero measure, in fact, depends upon $x \in E$. We are going to construct a modification of $(\xi(x), x \in E)$, such that it admits θ, and prove the corresponding convergence.

Let A be a dense countable subset of E. Since A is countable,

$$(\xi_n(x), x \in A) \underset{n\to\infty}{\longrightarrow} (\tilde{\xi}(x), x \in A) \tag{8}$$

pointwise almost surely. Consequently, using (3) and passing to the limit as $n \to \infty$ in the inequality

$$|\xi_n(x) - \xi_n(y)| \leq M_n\theta(|x-y|), \quad x, y \in A, \tag{9}$$

we obtain

$$|\tilde{\xi}(x) - \tilde{\xi}(y)| \leq M\theta(|x-y|) \tag{10}$$

for all $x, y \in A$ almost surely. In other words, realizations of $(\tilde{\xi}(x), x \in A)$ admit the modulus θ almost surely.

Define $(\xi(x), x \in E)$ by

$$\xi(x) \overset{\text{def}}{=} \inf_{y\in A} \left(\tilde{\xi}(y) + M\theta(|x-y|)\right). \tag{11}$$

We need to show that realizations of $(\xi(x), x \in E)$ also admit θ almost surely. First, suppose $x \in E$ and $y \in A$. The chain of inequalities

$$\begin{aligned} -M\theta(|x-y|) &\leq M \inf_{u\in A} (\theta(|x-u|) - \theta(|u-y|)) \\ &\leq \inf_{u\in A} \left(\tilde{\xi}(u) - \tilde{\xi}(y) + M\theta(|x-u|)\right) \leq M\theta(|x-y|), \end{aligned} \tag{12}$$

where the first one follows by subadditivity and monotonicity of θ, gives us

$$|\xi(x) - \tilde{\xi}(y)| = \left|\inf_{u\in A} \left(\tilde{\xi}(u) - \tilde{\xi}(y) + M\theta(|x-u|)\right)\right| \leq M\theta(|x-y|), \tag{13}$$

for all $x \in E$ and $y \in A$ almost surely. In particular, we see that $\xi(x) = \tilde{\xi}(x)$ for all $x \in A$ almost surely.

Next, for any $x, y \in E$ there exist sequences $\{x_k\} \subset A$ and $\{y_k\} \subset A$ such that $x_k \to x$ and $y_k \to y$ as $k \to \infty$. The triangle inequality and the formulas (10) and (13) yield

$$\begin{aligned} |\xi(x) - \xi(y)| &\leq |\xi(x) - \tilde{\xi}(x_k)| + |\tilde{\xi}(x_k) - \tilde{\xi}(y_k)| + |\tilde{\xi}(y_k) - \xi(y)| \\ &\leq M\big(\theta(|x-x_k|) + \theta(|x_k - y_k|) + \theta(|y_k - y|)\big). \end{aligned} \tag{14}$$

And passing to the limit as $k \to \infty$ and using the continuity of θ, we arrive at

$$|\xi(x) - \xi(y)| \leq M\theta(|x-y|) \tag{15}$$

for all $x, y \in E$ almost surely. This shows that realizations of $(\xi(x), x \in E)$ admit θ almost surely.

The final step is to establish the pointwise convergence in (5). Fix $x \in E$, and let $\{x_k\} \subset A$ be a sequence such that $x_k \to x$ as $k \to \infty$. The formulas (3) and (15), along with the triangle inequality, lead us to

$$\begin{aligned} |\xi_n(x) - \xi(x)| &\leq |\xi_n(x) - \xi_n(x_k)| + |\xi_n(x_k) - \xi(x_k)| + |\xi(x_k) - \xi(x)| \\ &\leq M_n\theta(|x-x_k|) + |\xi_n(x_k) - \xi(x_k)| + M\theta(|x_k - x|), \end{aligned} \tag{16}$$

which holds almost surely. Due to the continuity of θ, passing first to the limit superior as $n \to \infty$ and then to the limit as $k \to \infty$ yield (5) pointwise almost surely as desired. Due to the uniqueness of the limit it is also clear that $(\xi(x), x \in E)$ is a modification of the field $(\tilde{\xi}(x), x \in E)$. This completes the proof. □

Remark. *Fix $x_0 \in E$, and note that since E is bounded, the theorem also holds if one uses*

$$\tilde{M}_n \overset{\text{def}}{=} |\xi_n(x_0)| + \sup_{x \neq y} \frac{|\xi_n(x) - \xi_n(y)|}{\theta(|x-y|)}. \tag{17}$$

instead of M_n.

Indeed, passing to the supremum with respect to x in the elementary chain of inequalities, the terminal one being due to the boundedness of E and the continuity of θ,

$$\begin{aligned} |\xi_n(x)| \le \frac{|\xi_n(x) - \xi_n(x_0)|}{\theta(|x-x_0|)}\theta(|x-x_0|) + |\xi_n(x_0)| \\ \le \tilde{M}_n(1 + \theta(|x-x_0|)) \le \tilde{C}\tilde{M}_n \end{aligned} \tag{18}$$

gives

$$CM_n \le \tilde{M}_n \le M_n, \tag{19}$$

where $C = 1/\tilde{C} > 0$. Consequently, all estimates in the theorem carry over to the case of $\tilde{M}_n$.

A natural question arises whether one can guarantee a stronger mode of convergence in (5) and what assumptions are needed for this. We show below that provided further smoothness of the fields, indeed one can expect much more than just pointwise convergence. To alleviate unnecessary geometric complications we state the further result for one-dimensional domains only, namely $E = [a, b]$.

Denote the norm in the space of smooth functions $C^m(E)$ by

$$\|f\|_m = \sum_{l=0}^{m} \sup_x \left| f^{(l)}(x) \right|, \tag{20}$$

where $f^{(l)}$ is the l-th derivative of f and $f^{(0)} \overset{\text{def}}{=} f$. We have the following theorem.

Theorem 2. *Let $\{(\xi_n(x), x \in E)\}_{n \ge 0}$ be a sequence of stochastic processes such that their realizations are $C^{m+1}(E)$-smooth almost surely and such that realizations of their $(m+1)$-th derivatives admit a modulus of continuity θ almost surely. Set*

$$M_n \overset{\text{def}}{=} \|\xi_n\|_{m+1} + \sup_{x \neq y} \frac{\left|\xi_n^{(m+1)}(x) - \xi_n^{(m+1)}(y)\right|}{\theta(|x-y|)}, \tag{21}$$

and assume that $(\xi_n(x), \{\mathcal{F}_n\}_{n\geq 0})$ *is a martingale for every fixed* $x \in E$. *If*

$$\sup_n \mathbb{E}\,[M_n] < +\infty, \tag{22}$$

then there exist a random field $(\xi(x), x \in E)$ *with almost sure* $C^{m+1}(E)$*-smooth realizations and such that realizations of its* $(m+1)$*-th derivative admit the modulus of continuity* θ *almost surely; moreover*

$$\|\xi_n - \xi\|_m \underset{n\to\infty}{\longrightarrow} 0 \tag{23}$$

almost surely.

P r o o f. First, since $(\xi_n(x), \{\mathcal{F}_n\}_{n\geq 0})$ is a martingale for every fixed $x \in E$, we have

$$\mathbb{E}\left[\frac{\xi_n(x+h) - \xi_n(x)}{h}\,\middle|\,\mathcal{F}_{n_0}\right] = \frac{\xi_{n_0}(x+h) - \xi_{n_0}(x)}{h}, \quad n > n_0, \tag{24}$$

almost surely. The quantity M_n gives a bound for the absolute value of the expression inside the expectation, therefore due to (22) and due to the dominated convergence theorem one can pass to the limit as $h \to 0$ in (24). Consequently, $(\xi_n^{(1)}(x), \{\mathcal{F}_n\}_{n\geq 0})$ turns out to be a martingale for every fixed $x \in E$. Writing (24) for the derivatives and repeating the argument, one can easily show that $(\xi_n^{(l)}(x), \{\mathcal{F}_n\}_{n\geq 0})$ is a martingale for every $x \in E$ and for all $l = 1, \ldots, m+1$.

Also, it is clear that $(M_n, \{\mathcal{F}_n\}_{n\geq 0})$ is a submartingale which is bounded due to (22), thus

$$M_n \underset{n\to\infty}{\longrightarrow} M \tag{25}$$

for some random variable M with expectation $\mathbb{E}\,[M] < +\infty$.

We proceed further by induction. Consider the base case $m = 0$. Denote $\eta_n(x) = \xi_n^{(1)}(x)$ and observe that the assumptions of Theorem 1 are satisfied for (η_n) with the same modulus of continuity θ. Namely, one can see that if $\tilde{M}_n$ is defined by (3) of η_n and M_n is defined by (21), then $\tilde{M}_n = M_n - \|\xi_n\|_0 \leq M_n$. Thus, there is a stochastic process $(\eta(x), x \in E)$ such that its realizations admit θ almost surely, in particular they are almost sure continuous, and the convergence takes place

$$(\eta_n(x), x \in E) \underset{n\to\infty}{\longrightarrow} (\eta(x), x \in E) \tag{26}$$

pointwise almost surely.

Now we will construct the random field $(\xi(x), x \in E)$ such that $\xi^{(1)}(x) = \eta(x)$.

Since $(\xi_n(a), \{\mathcal{F}_n\}_{n\geq 0})$ is a bounded martingale, by Doob's convergence theorem we can find a random variable $\xi(a)$, $\mathbb{E}\,[\xi(a)] < +\infty$, such that

$$\xi_n(a) \underset{n\to\infty}{\longrightarrow} \xi(a) \tag{27}$$

almost surely.

Let us define $(\xi(x), x \in E)$ by

$$\xi(x) \stackrel{\text{def}}{=} \xi(a) + \int_a^x \eta(s)\, ds. \tag{28}$$

Clearly, $(\xi(x), x \in E)$ is $C^1(E)$-smooth almost surely and realizations of its first derivative admit θ almost surely. It remains to prove the convergence.

We have

$$\begin{aligned} \|\xi_n - \xi\|_0 &= \sup_x \left| \xi_n(a) - \xi(a) + \int_a^x (\eta_n(s) - \eta(s))\, ds \right| \\ &\leq |\xi_n(a) - \xi(a)| + \int_E |\eta_n(s) - \eta(s)|\; ds. \end{aligned} \tag{29}$$

Since η admits the modulus θ, it is bounded almost surely; also η_n are bounded uniformly in n almost surely since

$$\|\eta_n\|_0 \leq M_n \tag{30}$$

and M_n is an almost surely convergent sequence. Then by dominated convergence and using (26) and (27) we have

$$\|\xi_n - \xi\|_0 \underset{n\to\infty}{\longrightarrow} 0 \tag{31}$$

almost surely. This completes the proof of the base case.

Now let $k \geq 1$ and suppose that the claim holds for $m = k-1$. Then the inductive hypothesis hold for $\eta_n = \xi_n^{(1)}$ and the same θ. Thus, there exists a random field $(\eta(x), x \in E)$ we have that

$$\|\eta_n - \eta\|_{k-1} \underset{n\to\infty}{\longrightarrow} 0 \tag{32}$$

almost surely, η has almost sure $C^k(E)$-smooth realizations and the realizations of the k-th derivative admit θ almost surely.

Using the same definition for $(\xi(x), x \in E)$ as in (28), where $\xi(a)$ is as in (27), we see that realizations of this process are $C^{k+1}(E)$-smooth almost surely and the $(k+1)$-th derivative admits θ. To prove the convergence we notice that

$$\begin{aligned} \|\xi_n - \xi\|_k &= \|\xi_n - \xi\|_0 + \|\eta_n - \eta\|_{k-1} \\ &\le |\xi_n(a) - \xi(a)| + \sup_x \left| \int_a^x (\eta_n(s) - \eta(s))\, ds \right| + \|\eta_n - \eta\|_{k-1} \\ &\le |\xi_n(a) - \xi(a)| + \int_E |\eta_n(s) - \eta(s)|\, ds + \|\eta_n - \eta\|_{k-1}. \end{aligned} \tag{33}$$

Then, as $n \to \infty$, the first and the last term tend to zero almost surely by the formulas (27) and (32), and for the integral term we use the almost sure uniform convergence of $\eta_n(s) - \eta(s)$ to zero, also due to (32). Thus, we have

$$\|\xi_n - \xi\|_k \underset{n\to\infty}{\longrightarrow} 0. \tag{34}$$

This concludes the proof. □

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Azat Miftakhov
Moscow State University
Faculty of Mechanics and Mathematics
1 Leninskiye Gory 119991, Moscow, Russia
e-mail: miftakhov-af@tutanota.com